# Supplemental Material: Tuning excitation transport in a dissipative Rydberg ring

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## DERIVATION OF THE EFFECTIVE NON-HERMITIAN HAMILTONIAN

In this Supplemental Material, we derive the effective Hamiltonian from the full Hamiltonian where both $|\pm\rangle$ excitations are considered. According to Fig. 1, in the subspace spanned by the states $\{|\pm 000\rangle, |0\pm 00\rangle, |00\pm 0\rangle, |000\pm\rangle\}$, the system is described by the following full Hamiltonian

$$H_{\rm full}=\begin{pmatrix} \frac{\Delta}{2}-i\gamma & t_- & t_- & \frac{t_-}{8} & 0 & we^{2i\varphi_{12}} & we^{2i\varphi_{13}} & \frac{we^{2i\varphi_{14}}}{8} \\ t_- & \frac{\Delta}{2} & t_- & \frac{t_-}{3\sqrt{3}} & we^{2i\varphi_{21}} & 0 & we^{2i\varphi_{23}} & \frac{we^{2i\varphi_{24}}}{3\sqrt{3}} \\ t_- & t_- & \frac{\Delta}{2} & t_- & we^{2i\varphi_{31}} & we^{2i\varphi_{32}} & 0 & we^{2i\varphi_{34}} \\ \frac{t_-}{8} & \frac{t_-}{3\sqrt{3}} & t_- & \frac{\Delta}{2} & \frac{we^{2i\varphi_{41}}}{8} & \frac{we^{2i\varphi_{42}}}{3\sqrt{3}} & we^{2i\varphi_{43}} & 0 \\ 0 & we^{-2i\varphi_{12}} & we^{-2i\varphi_{13}} & \frac{we^{-2i\varphi_{14}}}{8} & -\frac{\Delta}{2}-i\gamma & t_+ & t_+ & \frac{t_+}{8} \\ we^{-2i\varphi_{21}} & 0 & we^{-2i\varphi_{23}} & \frac{we^{-2i\varphi_{24}}}{3\sqrt{3}} & t_+ & -\frac{\Delta}{2} & t_+ & \frac{t_+}{3\sqrt{3}} \\ we^{-2i\varphi_{31}} & we^{-2i\varphi_{32}} & 0 & we^{-2i\varphi_{34}} & t_+ & t_+ & -\frac{\Delta}{2} & t_+ \\ \frac{we^{-2i\varphi_{41}}}{8} & \frac{we^{-2i\varphi_{42}}}{3\sqrt{3}} & we^{-2i\varphi_{43}} & 0 & \frac{t_+}{8} & \frac{t_+}{3\sqrt{3}} & t_+ & -\frac{\Delta}{2} \end{pmatrix}. \tag{1}$$

The parameter $\Delta$ is the energy difference between the state $|+\rangle$ and the state $|-\rangle$. The hopping amplitudes are given by

$$t_\pm=\frac{\left|\langle\pm|\,\hat{d}^+\,|0\rangle\right|^2}{8\pi\epsilon_0 r_0^3}, \tag{2}$$

$$w=-\frac{3\,\langle +|\,\hat{d}^+\,|0\rangle\,\langle 0|\,\hat{d}^+\,|-\rangle}{8\pi\epsilon_0 r_0^3}. \tag{3}$$

Additionally, the van der Waals interaction is neglected in the parameter regime we consider, since it is much weaker compared to the resonant dipole-dipole interaction.

By defining the creation and annihilation operators on site $i$, $a_i^\dagger|0\rangle=|-\rangle_i$ and $b_i^\dagger|0\rangle=|+\rangle_i$, the Hamiltonian is expressed as

$$H_{\rm full}=-i\gamma\left(a_1^\dagger a_1+b_1^\dagger b_1\right)+\frac{\Delta}{2}\sum_i\left(a_i^\dagger a_i-b_i^\dagger b_i\right)+\sum_{i\neq j}\left(\frac{r_0}{r_{ij}}\right)^3\left(t_-a_i^\dagger a_j+t_+b_i^\dagger b_j+we^{2i\varphi_{ij}}a_i^\dagger b_j+we^{-2i\varphi_{ij}}b_i^\dagger a_j\right). \tag{4}$$

We work in the regime with $\Delta\gg t_\pm, w$, so that when we take $|0-00\rangle$ as the initial state, the system primarily remains in the subspace spanned by the states $\{|-000\rangle, |0-00\rangle, |00-0\rangle, |000-\rangle\}$. Adiabatically eliminating the $|+\rangle$ excitations thus lead to the effective Hamiltonian

$$H_{\rm eff}=\sum_i\varepsilon_i a_i^\dagger a_i+\sum_{ij}t_{ij}e^{i\phi_{ij}}a_i^\dagger a_j, \tag{5}$$

where $\varepsilon_i$ are the self-energy terms

$$\begin{aligned}
\varepsilon_1 &= -i\gamma + \frac{\Delta}{2} + \frac{w^2}{\Delta}\left[e^{2i(\varphi_{12}-\varphi_{21})} + e^{2i(\varphi_{13}-\varphi_{31})} + \frac{1}{64}e^{2i(\varphi_{14}-\varphi_{41})}\right] \\
&= -i\gamma + \frac{\Delta}{2} + \frac{w^2}{\Delta}\frac{129}{64}, \qquad (6)\\
\varepsilon_2 &= \frac{\Delta}{2} + \frac{w^2}{\Delta}\left[\frac{\Delta}{\Delta+i\gamma}e^{2i(\varphi_{21}-\varphi_{12})} + e^{2i(\varphi_{23}-\varphi_{32})} + \frac{1}{27}e^{2i(\varphi_{24}-\varphi_{42})}\right] \\
&= \frac{\Delta}{2} + \frac{w^2}{\Delta}\left(\frac{\Delta^2 - i\Delta\gamma}{\Delta^2+\gamma^2} + \frac{28}{27}\right), \qquad (7)\\
\varepsilon_3 &= \frac{\Delta}{2} + \frac{w^2}{\Delta}\left[\frac{\Delta}{\Delta+i\gamma}e^{2i(\varphi_{31}-\varphi_{13})} + e^{2i(\varphi_{32}-\varphi_{23})} + e^{2i(\varphi_{34}-\varphi_{43})}\right] \\
&= \frac{\Delta}{2} + \frac{w^2}{\Delta}\left(\frac{\Delta^2 - i\Delta\gamma}{\Delta^2+\gamma^2} + 2\right), \qquad (8)\\
\varepsilon_4 &= \frac{\Delta}{2} + \frac{w^2}{\Delta}\left[\frac{\Delta}{\Delta+i\gamma}\frac{1}{64}e^{2i(\varphi_{41}-\varphi_{14})} + \frac{1}{27}e^{2i(\varphi_{42}-\varphi_{24})} + e^{2i(\varphi_{43}-\varphi_{34})}\right] \\
&= \frac{\Delta}{2} + \frac{w^2}{\Delta}\left(\frac{\Delta^2 - i\Delta\gamma}{\Delta^2+\gamma^2}\frac{1}{64} + \frac{28}{27}\right), \qquad (9)
\end{aligned}$$

and $t_{ij}e^{i\phi_{ij}}$ are the hopping terms, which satisfy $t_{ji} = t_{ij}$ and $\phi_{ji} = -\phi_{ij}$, with

$$\begin{aligned}
t_{12}e^{i\phi_{12}} &= t_- + \frac{w^2}{\Delta}\left(e^{2i(\varphi_{23}-\varphi_{31})} + \frac{1}{24\sqrt{3}}e^{2i(\varphi_{24}-\varphi_{41})}\right) \\
&= t_- + \frac{w^2}{\Delta}\left[\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + \frac{1}{24\sqrt{3}}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right], \qquad (10)\\
t_{13}e^{i\phi_{13}} &= t_- + \frac{w^2}{\Delta}\left(e^{2i(\varphi_{32}-\varphi_{21})} + \frac{1}{8}e^{2i(\varphi_{34}-\varphi_{41})}\right) \\
&= t_- + \frac{w^2}{\Delta}\left[\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \frac{1}{8}\right], \qquad (11)\\
t_{14}e^{i\phi_{14}} &= \frac{t_-}{8} + \frac{w^2}{\Delta}\left(\frac{1}{3\sqrt{3}}e^{2i(\varphi_{42}-\varphi_{21})} + e^{2i(\varphi_{43}-\varphi_{31})}\right) \\
&= \frac{t_-}{8} + \frac{w^2}{\Delta}\left(-\frac{1}{3\sqrt{3}} + 1\right), \qquad (12)\\
t_{23}e^{i\phi_{23}} &= t_- + \frac{w^2}{\Delta}\left(\frac{\Delta}{\Delta+i\gamma}e^{2i(\varphi_{31}-\varphi_{12})} + \frac{1}{3\sqrt{3}}e^{2i(\varphi_{34}-\varphi_{42})}\right) \\
&= t_- + \frac{w^2}{\Delta}\left[\frac{\Delta^2 - i\Delta\gamma}{\Delta^2+\gamma^2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + \frac{1}{3\sqrt{3}}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right], \qquad (13)\\
t_{24}e^{i\phi_{24}} &= \frac{t_-}{3\sqrt{3}} + \frac{w^2}{\Delta}\left(\frac{1}{8}\frac{\Delta}{\Delta+i\gamma}e^{2i(\varphi_{41}-\varphi_{12})} + e^{2i(\varphi_{43}-\varphi_{32})}\right) \\
&= \frac{t_-}{3\sqrt{3}} + \frac{w^2}{\Delta}\left[\frac{1}{8}\frac{\Delta^2 - i\Delta\gamma}{\Delta^2+\gamma^2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right], \qquad (14)\\
t_{34}e^{i\phi_{34}} &= t_- + \frac{w^2}{\Delta}\left(\frac{1}{8}\frac{\Delta}{\Delta+i\gamma}e^{2i(\varphi_{41}-\varphi_{13})} + \frac{1}{3\sqrt{3}}e^{2i(\varphi_{42}-\varphi_{23})}\right) \\
&= t_- + \frac{w^2}{\Delta}\left[\frac{1}{8}\frac{\Delta^2 - i\Delta\gamma}{\Delta^2+\gamma^2} + \frac{1}{3\sqrt{3}}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right]. \qquad (15)
\end{aligned}$$