Supplemental Material: Predicted critical state based on invariance of the Lyapunov exponent in dual spaces

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I. DERIVATION OF LYAPUNOV EXPONENT γ IN POSITION SPACE

The Lyapunov exponent γ can be calculated by taking the product of the transfer matrix $T(\theta)$, namely multiplying the transfer matrix n times consecutively, which is written as

$$T_n(\theta)=\prod_{l=0}^{n-1}T(2\pi\alpha l+\theta)=\prod_{l=0}^{n-1}\begin{pmatrix} E-Vi\tan(2\pi\alpha l+\theta) & -1 \\ 1 & 0\end{pmatrix},$$

then Lyapunov exponent is $\ln||T_n(\theta)||/n$ as n tends to the infinite in the thermodynamic limit.

The method we use here to calculate Lyapunov exponent is the complexified phase approach, specifically by continuing the imaginary part of the phase ϵ, we focus on the new Lyapunov exponent, that is

$$T_n(\theta+i\epsilon)=\prod_{l=0}^{n-1}T(2\pi\alpha l+\theta+i\epsilon),$$

correspondingly, we get $\gamma(\epsilon)$ is $\lim_{n\to\infty}\ln||T_n(\theta+i\epsilon)||/n$.

Relying on Avila's global theory [1], if we can obtain Lyapunov exponent $\gamma(\epsilon)$ when ϵ is sufficiently large, then we can trace back to the specific Lyapunov exponent $\gamma(0)$ when $\epsilon=0$, namely the original Lyapunov exponent γ in position space.

Firstly, rewriting the transfer matrix

$$\begin{aligned} T(\theta)&=\begin{pmatrix} E-Vi\tan(2\pi\alpha l+\theta) & -1 \\ 1 & 0\end{pmatrix} \\ &=\sec(2\pi\alpha l+\theta)B(\theta)\end{aligned} \tag{1}$$

where

$$B(\theta)=\begin{pmatrix} E\cos(2\pi\alpha l+\theta)-Vi\sin(2\pi\alpha l+\theta) & -\cos(2\pi\alpha l+\theta) \\ \cos(2\pi\alpha l+\theta) & 0\end{pmatrix} \tag{2}$$

then, $\gamma(\epsilon)$ can be expressed as

$$\begin{aligned}\gamma_\epsilon(E)&=\lim_{n\to\infty}\ln||T_n(\theta+i\epsilon)||/n \\ &=\lim_{n\to\infty}\frac{1}{n}\int\ln|B_n(\theta+i\epsilon)|d\theta+\int\ln|\sec(\theta+i\epsilon)|d\theta. \\ &=\gamma_\epsilon^B(E)+\ln(2)-2\pi\epsilon,\end{aligned} \tag{3}$$

where

$$\gamma_\epsilon^B(E)=\lim_{n\to\infty}\frac{1}{n}\int\ln|B_n(\theta+i\varepsilon)|\,d\theta,$$

$$B_n(\theta+ic)=\prod_{l=0}^{n-1}B(2\pi\alpha l+\theta+i\epsilon).$$

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When ϵ tends to $+\infty$, a direct calculating result of $B(\theta+i\epsilon)$ is

$$B(\theta+i\epsilon)=-\frac{1}{2}e^{2\pi\epsilon}e^{i2\pi(\theta+\alpha)}\begin{pmatrix} E-V & -1 \\ 1 & 0 \end{pmatrix}+o(1). \tag{4}$$

Thus we get $\gamma^{+B}{}_{\epsilon}(E)=2\pi\epsilon+\ln|\frac{\sqrt{(E-V)^2\pm4}}{2}|-\ln(2)+o(1)$. As a function of ϵ, $\gamma^{B}_{\epsilon}(E)$ is a convex, piecewise linear function whose slope is an integer multiplied by 2π, hence it is concluded that when ϵ tends to infinity, we obtain$\gamma^{+B}{}_{\epsilon}(E)=2\pi\epsilon+\ln|\frac{\sqrt{(E-V)^2\pm4}}{2}|-\ln(2)$. And according to the equation(3), it leads that when ϵ is the very large positive number, $\gamma^{+}{}_{\epsilon}(E)=\gamma^{+B}{}_{\epsilon}(E)+\ln(2)-2\pi\epsilon=\ln|\frac{\sqrt{(E-V)^2\pm4}}{2}|$.

When ϵ tends to $-\infty$, a direct calculating result of $B(\theta+i\epsilon)$ is

$$B(\theta+i\epsilon)=-\frac{1}{2}e^{2\pi\epsilon}e^{i2\pi(\theta+\alpha)}\begin{pmatrix} E+V & -1 \\ 1 & 0 \end{pmatrix}+o(1). \tag{5}$$

Thus we get $\gamma^{-B}{}_{\epsilon}(E)=2\pi\epsilon+\ln|\frac{\sqrt{(E+V)^2\pm4}}{2}|-\ln(2)+o(1)$. As a function of ϵ, $\gamma^{B}_{\epsilon}(E)$ is a convex, piecewise linear function whose slope is an integer multiplied by 2π, hence it is concluded that when ϵ tends to infinity, we obtain $\gamma^{-B}{}_{\epsilon}(E)=2\pi\epsilon+\ln|\frac{\sqrt{(E+V)^2\pm4}}{2}|-\ln(2)$. And according to equation(3), it leads that when ϵ is the very large negative number, $\gamma^{-}{}_{\epsilon}(E)=\gamma^{-B}{}_{\epsilon}(E)+\ln(2)-2\pi\epsilon=\ln|\frac{\sqrt{(E+V)^2\pm4}}{2}|$.

Since γ_ϵ is a convex function in two semilinear $(0,+\infty)$ and $(-\infty,0)$, it is linear in the cross section and the slope is an integer multiplied by the 2π integer, Lyapunov exponent is

$$\gamma_\varepsilon(E)=\begin{cases} \gamma^{+}_{\varepsilon}(E) & \varepsilon>0, \\ \gamma^{+}_{\varepsilon}(E)+2\varepsilon & \frac{\gamma^{-}_{\varepsilon}(E)-\gamma^{+}_{\varepsilon}(E)}{2}<\varepsilon<0, \\ \gamma^{-}_{\varepsilon}(E) & \varepsilon<\frac{\gamma^{-}_{\varepsilon}(E)-\gamma^{+}_{\varepsilon}(E)}{2}, \end{cases}$$

the relationship between the left and right limit conditions is $\gamma^{+}_{\varepsilon}(E)>\gamma^{-}_{\varepsilon}(E)$ for any given value of ε.

Similarly, if the large and small relationship between the left and right limits is $\gamma^{+}_{\varepsilon}(E)<\gamma^{-}_{\varepsilon}(E)$, Lyapunov exponent is

$$\text{then } \gamma_\varepsilon(E)=\begin{cases} \gamma^{-}_{\varepsilon}(E) & \varepsilon<0, \\ \gamma^{-}_{\varepsilon}(E)-2\varepsilon & 0<\varepsilon<\frac{\gamma^{-}_{\varepsilon}(E)-\gamma^{+}_{\varepsilon}(E)}{2}, \\ \gamma^{+}_{\varepsilon}(E) & \varepsilon>\frac{\gamma^{-}_{\varepsilon}(E)-\gamma^{+}_{\varepsilon}(E)}{2}. \end{cases}$$

Summarizing the above conclusions, Lyapunov exponent in position space is $\gamma=\max\{\gamma^{+}_{\varepsilon}(E),\gamma^{-}_{\varepsilon}(E)\}$, which is

$$\gamma(E)=\max\Big\{\operatorname{arcosh}\frac{|E+V+2|+|E+V-2|}{4}, \operatorname{arcosh}\frac{|E-V+2|+|E-V-2|}{4}\Big\}. \tag{6}$$

II. DERIVATION OF LYAPUNOV EXPONENT γ_m IN MOMENTUM SPACE

In the main text, utilizing Fourier transform, the initial wave function solution has been obtained. Relying on Jensen's formula [2], our calculation supposes that f is an analytic function, $a_1,a_2,\ldots,a_n$ are the zeros of f in the interior of the unit disc of the complex plane, and $f(0)\neq0$. Then, we have the following equality

$$\ln|f|=\sum_{k=1}^{n}\ln(|a_k|)+\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f\left(e^{i\theta}\right)\right|d\theta. \tag{7}$$

Combining the initial wave function solution, the expression of Lyapunov exponent can be written as

$$\gamma_m(E)=\lim_{k\to\infty}\frac{1}{k-k_0}\ln\left|\frac{f_k}{f_{k_0}}\right|=\int[\ln g^{(1)}-\ln g^{(2)}]d\theta, \tag{8}$$

where $g^{(1)} = |-2\cos(2\pi\theta) + V + E|$, $g^{(2)} =| 2\cos(2\pi\theta)+ V - E |$.

Then, the first term of the rightmost side of γ_m can be written as

$$\begin{aligned}\int_0^1 \ln g^{(1)} d\theta &= \int_0^1 \ln|-2\cos(2\pi\theta) + V + E|d\theta \\ &= \int_0^1 \ln|-e^{i2\pi\theta} - e^{-i2\pi\theta} + V + E|d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \ln|e^{i2\theta} + 1 - (V+E)e^{i\theta}|d\theta.\end{aligned} \tag{9}$$

Applying Jensen's formula [2], the integral calculation of Eq. S8 can be transformed to the calculation of roots of Eq. S9 in the unit disc, let $x = e^{i\theta}$,

$$x^2 + 1 - (V+E)x = 0. \tag{10}$$

After some mathematical calculations, we obtain

$$\begin{cases} \text{when } |V+E| < 2, \int_0^1 \ln g^{(1)} d\theta = 0, \\ \text{when } |V+E| > 2, \int_0^1 \ln g^{(1)} d\theta = \ln\frac{|E+V| + \sqrt{(E+V)^2-4}}{2}. \end{cases} \tag{11}$$

Under the similar process, with regard to the second term of γ_m, we also obtain

$$\begin{cases} \text{when } |V-E| < 2, \int_0^1 \ln g^{(2)} d\theta = 0, \\ \text{when } |V-E| > 2, \int_0^1 \ln g^{(2)} d\theta = \ln\frac{|E-V| + \sqrt{(E-V)^2-4}}{2}. \end{cases} \tag{12}$$

Interesting, when both $|V+E| < 2$ and $|V-E| < 2$ hold, we have

$$\gamma_m(E) = \int [\ln g^{(1)} - \ln g^{(2)}] d\theta = 0. \tag{13}$$

When both $|V+E| < 2$ and $|V-E| > 2$ hold, then

$$\gamma_m(E) = \int [\ln g^{(1)} - \ln g^{(2)}] d\theta = 0 - \ln\frac{|E-V| + \sqrt{(E-V)^2-4}}{2} < 0. \tag{14}$$

When both $|V+E| > 2$ and $|V-E| < 2$ hold, then

$$\gamma_m(E) = \int [\ln g^{(1)} - \ln g^{(2)}] d\theta = \ln\frac{|E+V| + \sqrt{(E+V)^2-4}}{2} - 0 > 0. \tag{15}$$

When both $|V+E| > 2$ and $|V-E| > 2$ hold, then

$$\gamma_m(E) = \int [\ln g^{(1)} - \ln g^{(2)}] d\theta = \ln\frac{|E+V| + \sqrt{(E+V)^2-4}}{2} - \ln\frac{|E-V| + \sqrt{(E-V)^2-4}}{2} \neq 0. \tag{16}$$

To sum up the above calculations, only when both $|V+E| < 2$ and $|V-E| < 2$ hold, namely the eigenvalues $E \in [V-2, 2-V]$, γ_m is equal to 0.

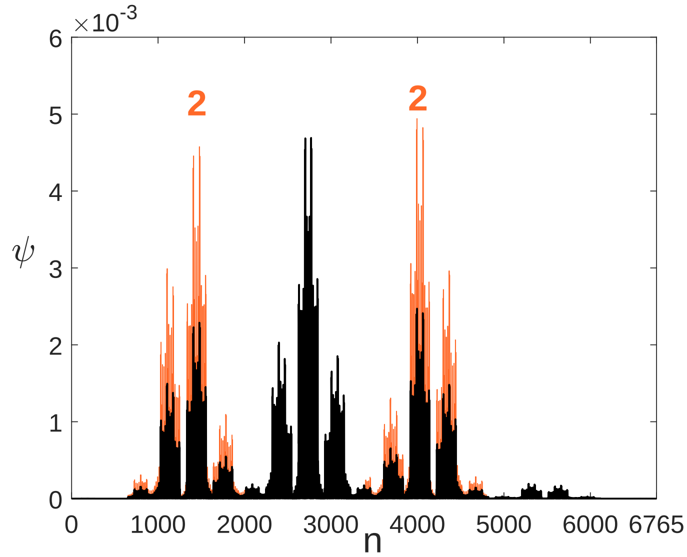

Figure 1: (Color online) The black curve represents wave function of $E = 0.5$ with the parameter $V = 1$. The red curves represent three wave function peaks after magnifying. It clearly shows that the scaled two smaller peaks are very similar to the largest peak after twice magnification. The total number of sites is set to be $L = 6765$.

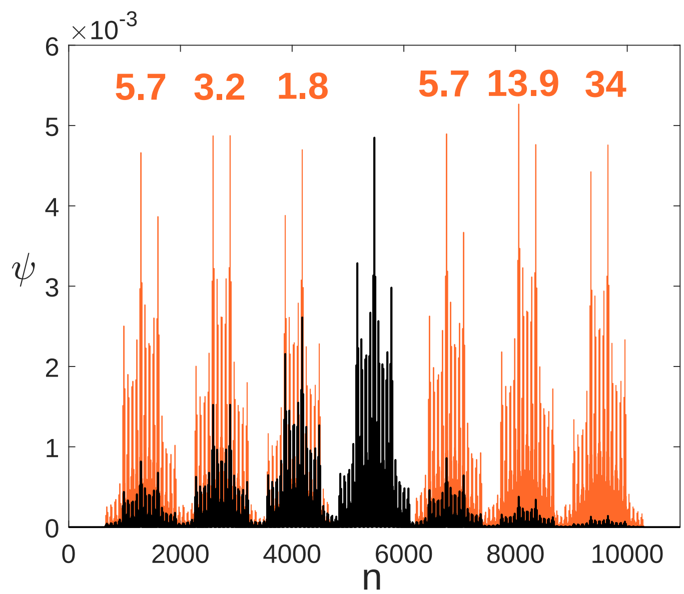

Figure 2: (Color online) The black curve represents wave function of $E = -1$ with the parameter $V = 1$. The red curves represent three wave function peaks after magnifying. The scaled multiples of three left small peaks are 5.7, 3.2 and 1.8 in turn; the scaled multiples of three right small peaks are 5.7, 13.9 and 34 in turn. The total number of sites is set to be $L = 10946$.

III. MORE NUMERICAL VERIFICATION

In this section, we provide more numerical validation to strengthen the credibility of our theoretical prediction. We have calculated the wave functions of different energy levels at the same size, and the same energy level at different sizes. The numerical results are as expected, and the corresponding eigenstates dispaly self-similarity. As shown in Fig. S1 and Fig. S2, it clearly illusrates that the scaled smaller peaks are very similar to the largest peak. More interesting, in Fig. S2, three left small peaks satisfy the scale invariance with a certain multiple $5.7/3.2 \approx 3.2/1.8 \approx 1.78$; whereas three right small peaks satisfy the scale invariance with a certain multiple $13.9/5.7 \approx 34/13.9 \approx 2.44$. The different scaling factors indicate that there is more than one fractal structure in the wave function, which exactly corresponds to the multifractal theory in the introduction. We also calculate the corresponding eigenstate in momentum space, which is shown in Fig. S3.

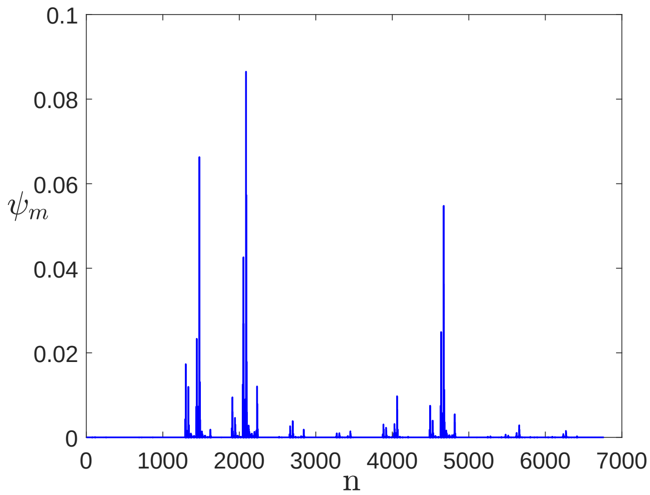

Figure 3: (Color online) The blue curve represents wave function of $E = -1$ obtained from Eq. (**??**) with the parameter $V = 1$ in momentum space. It clearly shows the self-similarity property, indicating the corresponding wave function is critical state. The total number of sites is set to be $L = 6765$.

[1] A. Avila, Global theory of one-frequency Schrödinger operators, Acta. Math. **1**, 215, (2015).
[2] J. Jensen, Sur un nouvel et important théorème de la théorie des fonctions, Acta Mathematica **1**,22, (1899).