Supplementary Material: Modulation of steady-state heat transport in the dissipative multi-mode qubit-photon system

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Cycle fluxes with weak qubit-resonator interaction in three-mode qubit-resonator model

In the weak qubit-resonator interaction regime, i.e., $\Lambda_i/\Omega_i \ll 1$, the coherent state overlap coefficient is simplified as

$$D_{n_i n_i'}(2\Lambda_i/\Omega_i) \;\approx\; (-1)^{n_i}[\delta_{n_i,n_i'} + 2\Lambda_i/\Omega_i\sqrt{n_i+1}\delta_{n_i,n_i'-1} - 2\Lambda_i/\Omega_i\sqrt{n_i}\delta_{n_i,n_i'+1}]. \tag{S1}$$

Hence, the rate involved with the q-th reservoir is approximated as $\Gamma_{q,\pm}(\Delta^{\overline{\sigma},\sigma}_{\mathbf{n}',\mathbf{n}}) \approx \Gamma^{(0)}_{q,\pm}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}'}) + \Gamma^{(1)}_{q,\pm}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}'})$, where the zeroth-order is given by

$$\Gamma^{(0)}_{q,\pm}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}'}) = \kappa_q(\pm\varepsilon)\delta_{\mathbf{n},\mathbf{n}'}, \tag{S2}$$

with $\kappa_q(\pm\varepsilon) = \gamma_q(\pm\varepsilon)n_q(\pm\varepsilon)$. And the first-order term is given by

$$\Gamma^{(1)}_{q,\pm}(\Delta^{\overline{\sigma},\sigma}_{\mathbf{n}',\mathbf{n}}) = [\Gamma^{\pm}_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n}+\mathbf{I}_1,\mathbf{n}}) + \Gamma^{\pm}_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n}-\mathbf{I}_1,\mathbf{n}}) + \Gamma^{\pm}_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n}+\mathbf{I}_3,\mathbf{n}}) + \Gamma^{\pm}_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n}-\mathbf{I}_3,\mathbf{n}}) + \Gamma^{\pm}_q(\Delta^{\downarrow,\uparrow}_{\mathbf{n}+\mathbf{I}_1,\mathbf{n}}) + \Gamma^{\pm}_q(\Delta^{\downarrow,\uparrow}_{\mathbf{n}+\mathbf{I}_3,\mathbf{n}})], \tag{S3}$$

with the components specified as

$$\Gamma^{\pm}_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n}+\mathbf{I}_1,\mathbf{n}}) = \kappa^{\pm}_q(\varepsilon+\Omega_1)(2\Lambda_1/\Omega_1)^2 n_1'\delta_{n_1,n_1'-1}\delta_{n_2,n_2'}\delta_{n_3,n_3'}, \tag{S4a}$$
$$\Gamma^{\pm}_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n}-\mathbf{I}_1,\mathbf{n}}) = \Theta(\varepsilon-\Omega_1)\kappa^{\pm}_q(\varepsilon-\Omega_1)(2\Lambda_1/\Omega_1)^2 n_1'\delta_{n_1,n_1'+1}\delta_{n_2,n_2'}\delta_{n_3,n_3'}, \tag{S4b}$$
$$\Gamma^{\pm}_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n}+\mathbf{I}_3,\mathbf{n}}) = \kappa^{\pm}_q(\varepsilon+\Omega_3)(2\Lambda_3/\Omega_3)^2 n_3'\delta_{n_1,n_1'}\delta_{n_2,n_2'}\delta_{n_3,n_3'-1}, \tag{S4c}$$
$$\Gamma^{\pm}_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n}-\mathbf{I}_3,\mathbf{n}}) = \Theta(\varepsilon-\Omega_3)\kappa^{\pm}_q(\varepsilon-\Omega_3)(2\Lambda_3/\Omega_3)^2 n_3'\delta_{n_1,n_1'}\delta_{n_2,n_2'}\delta_{n_3,n_3'+1}, \tag{S4d}$$
$$\Gamma^{\pm}_q(\Delta^{\downarrow,\uparrow}_{\mathbf{n}+\mathbf{I}_1,\mathbf{n}}) = \Theta(\Omega_1-\varepsilon)\kappa^{\pm}_q(\Omega_1-\varepsilon)(2\Lambda_1/\Omega_1)^2 n_1'\delta_{n_1,n_1'-1}\delta_{n_2,n_2'}\delta_{n_3,n_3'}, \tag{S4e}$$
$$\Gamma^{\pm}_q(\Delta^{\downarrow,\uparrow}_{\mathbf{n}+\mathbf{I}_3,\mathbf{n}}) = \Theta(\Omega_3-\varepsilon)\kappa^{\pm}_q(\Omega_3-\varepsilon)(2\Lambda_3/\Omega_3)^2 n_3'\delta_{n_1,n_1'}\delta_{n_2,n_2'}\delta_{n_3,n_3'-1}. \tag{S4f}$$

Then, if we reexpress the quantum master equation as $d|\mathbf{P}\rangle/dt = \mathcal{L}|\mathbf{P}\rangle$ with $\mathcal{L} \approx \mathcal{L}^{(0)} + (\frac{\lambda}{\omega_a})^2\mathcal{L}^{(1)}$ and $|\mathbf{P}\rangle \approx |\mathbf{P}^{(0)}\rangle + (\frac{\lambda}{\omega_a})^2|\mathbf{P}^{(1)}\rangle$, the zeroth-order steady state solution is $\mathcal{L}^{(0)}|\mathbf{P}^{(0)}\rangle = 0$, and the first-order solution is $\mathcal{L}^{(0)}|\mathbf{P}^{(1)}\rangle + \mathcal{L}^{(1)}|\mathbf{P}^{(0)}\rangle = 0$. Hence, the zeroth-order population dynamics is given by

$$\begin{aligned}\frac{dP^{(0)}_{\mathbf{n},\downarrow}}{dt} =& \sum_i[\Gamma^{-}_{r_i}(\Delta^{\downarrow,\downarrow}_{\mathbf{n}+\mathbf{I}_i,\mathbf{n}})P^{(0)}_{\mathbf{n}+\mathbf{I}_i,\downarrow} - \Gamma^{+}_{r_i}(\Delta^{\downarrow,\downarrow}_{\mathbf{n}+\mathbf{I}_i,\mathbf{n}})P^{(0)}_{\mathbf{n},\downarrow}] + \sum_i[\Gamma^{+}_{r_i}(\Delta^{\downarrow,\downarrow}_{\mathbf{n},\mathbf{n}-\mathbf{I}_i})P^{(0)}_{\mathbf{n}-\mathbf{I}_i,\downarrow} - \Gamma^{-}_{r_i}(\Delta^{\downarrow,\downarrow}_{\mathbf{n},\mathbf{n}-\mathbf{I}_i})P^{(0)}_{\mathbf{n},\downarrow}]\\ &+ [\Gamma^{(0)}_{q,-}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}})P^{(0)}_{\mathbf{n},\uparrow} - \Gamma^{(0)}_{q,+}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}})P^{(0)}_{\mathbf{n},\downarrow}],\end{aligned} \tag{S5a}$$

$$\begin{aligned}\frac{dP^{(0)}_{\mathbf{n},\uparrow}}{dt} =& \sum_i[\Gamma^{-}_{r_i}(\Delta^{\uparrow,\uparrow}_{\mathbf{n}+\mathbf{I}_i,\mathbf{n}})P^{(0)}_{\mathbf{n}+\mathbf{I}_i,\uparrow} - \Gamma^{+}_{r_i}(\Delta^{\uparrow,\uparrow}_{\mathbf{n}+\mathbf{I}_i,\mathbf{n}})P^{(0)}_{\mathbf{n},\uparrow}] + \sum_i[\Gamma^{+}_{r_i}(\Delta^{\uparrow,\uparrow}_{\mathbf{n},\mathbf{n}-\mathbf{I}_i})P^{(0)}_{\mathbf{n}-\mathbf{I}_i,\uparrow} - \Gamma^{-}_{r_i}(\Delta^{\uparrow,\uparrow}_{\mathbf{n},\mathbf{n}-\mathbf{I}_i})P^{(0)}_{\mathbf{n},\uparrow}]\\ &- [\Gamma^{(0)}_{q,-}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}})P^{(0)}_{\mathbf{n},\uparrow} - \Gamma^{(0)}_{q,+}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}})P^{(0)}_{\mathbf{n},\downarrow}].\end{aligned} \tag{S5b}$$

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This leads to the zeroth order steady-state populations

$$P^{(0)}_{\mathbf{n},\uparrow} = \frac{1}{e^{\beta_q\varepsilon}+1}\Pi_i[(1-e^{-\beta_{r_i}\Omega_i})e^{-n_i\beta_{r_i}\Omega_i}], \tag{S6a}$$

$$P^{(0)}_{\mathbf{n},\downarrow} = \frac{e^{\beta_q\varepsilon}}{e^{\beta_q\varepsilon}+1}\Pi_i[(1-e^{-\beta_{r_i}\Omega_i})e^{-n_i\beta_{r_i}\Omega_i}]. \tag{S6b}$$

While the first-order populations dynamics are given by

$$\begin{aligned}(\frac{\lambda}{\omega_a})^2\frac{dP^{(1)}_{\mathbf{n},\uparrow}}{dt} =&(\frac{\lambda}{\omega_a})^2\sum_i[\Gamma^-_{r_i}(\Delta^{\uparrow,\uparrow}_{\mathbf{n+I}_i,\mathbf{n}})P^{(1)}_{\mathbf{n+I}_i,\uparrow}-\Gamma^+_{r_i}(\Delta^{\uparrow,\uparrow}_{\mathbf{n+I}_i,\mathbf{n}})P^{(1)}_{\mathbf{n},\uparrow}]+(\frac{\lambda}{\omega_a})^2\sum_i[\Gamma^+_{r_i}(\Delta^{\uparrow,\uparrow}_{\mathbf{n},\mathbf{n-I}_i})P^{(1)}_{\mathbf{n-I}_i,\uparrow}-\Gamma^-_{r_i}(\Delta^{\uparrow,\uparrow}_{\mathbf{n},\mathbf{n-I}_i})P^{(1)}_{\mathbf{n},\uparrow}]\\&-[\Gamma^{(0)}_{q,-}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}})P^{(1)}_{\mathbf{n},\uparrow}-\Gamma^{(0)}_{q,+}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}})P^{(1)}_{\mathbf{n},\downarrow}]-\sum_i[\Gamma^-_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n-I}_i})P^{(0)}_{\mathbf{n},\uparrow}-\Gamma^+_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n-I}_i})P^{(0)}_{\mathbf{n-I}_i,\downarrow}]\\&-\sum_i[\Gamma^-_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n+I}_i})P^{(0)}_{\mathbf{n},\uparrow}-\Gamma^+_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n+I}_i})P^{(0)}_{\mathbf{n+I}_i,\downarrow}]+\sum_i[\Gamma^-_q(\Delta^{\downarrow,\uparrow}_{\mathbf{n+I}_i,\mathbf{n}})P^{(0)}_{\mathbf{n+I}_i,\downarrow}-\Gamma^+_q(\Delta^{\downarrow,\uparrow}_{\mathbf{n+I}_i,\mathbf{n}})P^{(0)}_{\mathbf{n},\uparrow}],\end{aligned} \tag{S7a}$$

$$\begin{aligned}(\frac{\lambda}{\omega_a})^2\frac{dP^{(1)}_{\mathbf{n},\downarrow}}{dt} =&(\frac{\lambda}{\omega_a})^2\sum_i[\Gamma^-_{r_i}(\Delta^{\downarrow,\downarrow}_{\mathbf{n+I}_i,\mathbf{n}})P^{(1)}_{\mathbf{n+I}_i,\downarrow}-\Gamma^+_{r_i}(\Delta^{\downarrow,\downarrow}_{\mathbf{n+I}_i,\mathbf{n}})P^{(1)}_{\mathbf{n},\downarrow}]+(\frac{\lambda}{\omega_a})^2\sum_i[\Gamma^+_{r_i}(\Delta^{\downarrow,\downarrow}_{\mathbf{n},\mathbf{n-I}_i})P^{(1)}_{\mathbf{n-I}_i,\downarrow}-\Gamma^-_{r_i}(\Delta^{\downarrow,\downarrow}_{\mathbf{n},\mathbf{n-I}_i})P^{(1)}_{\mathbf{n},\downarrow}]\\&+[\Gamma^{(0)}_{q,-}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}})P^{(1)}_{\mathbf{n},\uparrow}-\Gamma^{(0)}_{q,+}(\Delta^{\uparrow,\downarrow}_{\mathbf{n},\mathbf{n}})P^{(1)}_{\mathbf{n},\downarrow}]+\sum_i[\Gamma^-_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n+I}_i,\mathbf{n}})P^{(0)}_{\mathbf{n+I}_i,\uparrow}-\Gamma^+_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n+I}_i,\mathbf{n}})P^{(0)}_{\mathbf{n},\downarrow}]\\&+\sum_i[\Gamma^-_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n-I}_i,\mathbf{n}})P^{(0)}_{\mathbf{n-I}_i,\uparrow}-\Gamma^+_q(\Delta^{\uparrow,\downarrow}_{\mathbf{n-I}_i,\mathbf{n}})P^{(0)}_{\mathbf{n},\downarrow}]-\sum_i[\Gamma^-_q(\Delta^{\downarrow,\uparrow}_{\mathbf{n},\mathbf{n-I}_i})P^{(0)}_{\mathbf{n},\downarrow}-\Gamma^+_q(\Delta^{\downarrow,\uparrow}_{\mathbf{n},\mathbf{n-I}_i})P^{(0)}_{\mathbf{n-I}_i,\uparrow}],\end{aligned} \tag{S7b}$$

Finally, the steady state heat current is expressed as

$$J_q \approx (C_a+C_b+C_c), \tag{S8}$$

where three cycle components are specified as

$$C_a=\sum_{i=1,3}(4\Lambda_i^2/\Omega_i)\Theta(\varepsilon+\Omega_i)\frac{\gamma_q(\varepsilon+\Omega_i)}{2n_q(\varepsilon)+1}[(1+n_q(\varepsilon+\Omega_i))n_q(\varepsilon)n_{r_i}(\Omega_i)-n_q(\varepsilon+\Omega_i)(1+n_q(\varepsilon))(1+n_{r_i}(\Omega_i))], \tag{S9a}$$

$$C_b=\sum_{i=1,3}(4\Lambda_i^2/\Omega_i)\Theta(\varepsilon-\Omega_i)\frac{\gamma_q(\varepsilon-\Omega_i)}{2n_q(\varepsilon)+1}[n_q(\varepsilon-\Omega_i)(1+n_q(\varepsilon))n_{r_i}(\Omega_i)-(1+n_q(\varepsilon-\Omega_i))n_q(\varepsilon)(1+n_{r_i}(\Omega_i))], \tag{S9b}$$

$$C_c=\sum_{i=1,3}(4\Lambda_i^2/\Omega_i)\Theta(\Omega_i-\varepsilon)\frac{\gamma_q(\Omega_i-\varepsilon)}{2n_q(\varepsilon)+1}[(1+n_q(\Omega_i-\varepsilon))(1+n_q(\varepsilon))n_{r_i}(\Omega_i)-n_q(\Omega_i-\varepsilon)n_q(\varepsilon)(1+n_{r_i}(\Omega_i))]. \tag{S9c}$$