Supplementary Materials for
"Wave-Particle Duality via Quantum Fisher Information"

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QFI FOR QUBIT

The classical Fisher information, originating from the statistical inference, is a way of measuring the amount of information that an observable random variable $\hat{X}$ carries about an unknown parameter θ. Suppose that $\{p_x(\theta)\}_{x=1}^N$ is the probability distribution conditioned on the fixed value of the parameter $\theta = \theta^*$ with measurement outcomes $\{1 \le x \le N\}$. The classical Fisher information is defined as

$$I_\theta[\{p_x(\theta)\}] = \sum_{x=1}^{N} \frac{(\dot{p}_x(\theta))^2}{p_x(\theta)}, \tag{1}$$

whose inverse provides the lower bound for the variance of an unbiased estimation of θ.

In quantum metrology, we want to estimate an unknown parameter θ that is encoded in a given quantum state $\rho(\theta)$. For that purpose we perform some measurement, most generally a POVM $\{M_x \ge 0, \sum_x M_x = I\}$, on the system yielding a classical statistics $p_x(\theta) = \mathrm{Tr}(\rho(\theta)M_x)$ that is dependent of the parameter θ with the precision of estimation quantified by classical Fisher information as defined in Eq.(1). By maximizing over all possible measurements we obtain the quantum fisher information (QFI) [1]

$$F_\theta[\rho] = \max_{\{M_x\}} I[\{p_x(\theta)\}] = \sum_{jk} \frac{2|\langle\phi_j|\,\dot{\rho}(\theta)\,|\phi_k\rangle|^2}{\lambda_j + \lambda_k}, \tag{2}$$

with $\{\lambda_j, \phi_j\}$ being the eigenvalues and eigenstates of $\rho(\theta)$ and $\dot{\rho}(\theta) = d\rho/d\theta$.

We note that QFI is non-increasing under any completely positive map $\mathcal{E}$ (e.g., partial trace). Let $\{E_k\}$ be the Kraus operators for $\mathcal{E}$, i.e.,

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

and then $\tilde{M}_x = \sum_k E_k^\dagger M_x E_k$ is still a POVM so we have

$$q_x(\theta) = \mathrm{Tr}[M_x \mathcal{E}(\rho)] = \mathrm{Tr}[\tilde{M}_x \rho] = \tilde{p}_x(\theta)$$

and

$$F_\theta[\mathcal{E}(\rho)] = I_\theta[q_x(\theta)] = I_\theta[\tilde{p}_x(\theta)] \le F_\theta[\rho]$$

where the first equality is because we have chosen M_x to be optimal for estimating θ on state $\mathcal{E}(\rho)$ and the last inequality is because the measurement $\tilde{M}_x$ might not be optimal for estimating θ on ρ.

In the case of a qubit state ρ with a Bloch vector $\vec{r} = \mathrm{Tr}(\vec{\sigma}\rho)$ with its dependence of θ given by $\dot{\rho} = \partial\rho/\partial\theta = (\partial\vec{r}/\partial\theta)\cdot\vec{\sigma}/2 = \dot{\vec{r}}\cdot\vec{\sigma}/2$, the QFI reads

$$F_\theta[\rho] = |\dot{\vec{r}}|^2 + \frac{(\dot{\vec{r}}\cdot\vec{r})^2}{1-\vec{r}^2} = \frac{|\dot{\vec{r}}|^2 - (\dot{\vec{r}}\times\vec{r})^2}{1-\vec{r}^2}, \tag{3}$$

which is attained by an ideal projective measurement along the direction (in the Bloch sphere representation)

$$\vec{m} \propto \dot{\vec{r}} + \frac{(\dot{\vec{r}}\cdot\vec{r})}{1-\vec{r}^2}\vec{r} \propto \dot{\vec{r}} - (\dot{\vec{r}}\times\vec{r})\times\vec{r}. \tag{4}$$

DERIVATION OF MAIN RESULTS EQS. (5&6) IN THE MAIN TEXT

We express the initial state of a two-level system in the Bloch representation as

$$\rho = \frac{\hat{I} + x(\sigma_x\cos\Phi + \sigma_y\sin\Phi) + z\sigma_z}{2}. \tag{5}$$

After its interaction with the detector plus environment $P_0 = e_0$, the state of the total system reads

$$\rho_f = U(\rho\otimes P_0)U^\dagger = \sum_{a,b=0}^{1} \rho_{ab}\,|a\rangle\langle b| \otimes |e_a\rangle\langle e_b|, \tag{6}$$

with $\rho_{ab} = \langle a|\,\rho\,|b\rangle$, and $a, b = 0, 1$. We then have

$$\rho_S = \mathrm{Tr}_{DE}\rho_f = \sum_{a,b=0}^{1} \rho_{ab}\langle e_a|e_b\rangle\,|a\rangle\langle b| = \frac{1 + \vec{r}\cdot\vec{\sigma}}{2}, \tag{7}$$

with $\vec{r} = (\kappa x\cos(\Phi+\delta), \kappa x\sin(\Phi+\delta), z)$, and $\kappa = |\langle e_0|e_1\rangle|$, $\langle e_0|e_1\rangle = \kappa e^{i\delta}$. We then obtain QFI

$$F_\Phi[\rho_S] = \kappa^2 x^2 = |\langle e_0|e_1\rangle|^2 x^2 \tag{8}$$

from the optimal measurement $\hat{M}_\Phi = \vec{m}_\Phi\cdot\vec{\sigma}$ on the system state with $\vec{m}_\Phi \propto (-\sin\Phi, \cos\Phi, 0)$.

The optimal estimation of z, on the other hand, is achieved by a suitable measurement on the detector, which might be a high dimensional quantum system for which the QFI does not admit a compact analytical expression. However due to the non-increasing property of QFI under partial trace, we have

$$F_z[\rho_D] \le F_z[\rho_{DE}], \tag{9}$$

so that we have only to calculate the QFI of estimating parameter z on the state of detector plus environment

$$\rho_{DE} = \mathrm{Tr}_S \rho_f = \sum_{a,b=0}^{1} \rho_{ab} \left|e_a\right\rangle \left\langle e_a\right| . \tag{10}$$

The state ρ_{DE} is of rank two and its QFI can be calculated analytically. In fact it can be regarded effectively as a state of a two-level system spanned by $\{|e_0\rangle, |e_1\rangle\}$. Denoting by $\vec{\Sigma}$ as Pauli matrices of this effective qubit, we have Bloch vectors $\vec{s_a} = \langle e_a|\vec{\Sigma}|e_a\rangle$ and

$$\rho_{DE} = \frac{1+\vec{R}\cdot\vec{\Sigma}}{2}, \quad \vec{R} = \frac{\vec{s_0}+\vec{s_1}+z(\vec{s_0}-\vec{s_1})}{2}, \tag{11}$$

with

$$\frac{\partial \vec{R}}{\partial z} = \frac{\vec{s_0}-\vec{s_1}}{2}.$$

According to Eq.(3), we obtain the upper bound

$$F_z[\rho_{DE}] = \frac{1-\vec{s_0}\cdot\vec{s_1}}{2(1-z^2)} = \frac{1-|\langle e_0|e_1\rangle|^2}{1-z^2}, \tag{12}$$

Putting everything together, Eqs.(9,8), $P^2 = z^2$, and $V^2 = x^2$, we have

$$(1-P^2)F_z[\rho_D] + \frac{F_\Phi[\rho_S]}{V^2} \le (1-z^2)F_z[\rho_{DE}] + \frac{F_\Phi[\rho_S]}{x^2} \le 1$$

Moreover since $F_\varphi[\rho_S] \ge 0$ and $x^2 \le 1-z^2$, we have

$$\begin{aligned} F_z[\rho_D]F_\varphi[\rho_S] &\le F_z[\rho_{DE}]F_\varphi[\rho_S] \\ &\le (1-|\langle e_0|e_1\rangle|^2)|\langle e_0|e_1\rangle|^2 \le \frac{1}{4}. \end{aligned}$$

We have $F_z[\rho_D]F_\varphi[\rho_S] = \frac{1}{4}$ iff $|\langle e_0|e_1\rangle|^2 = 1/2$.

DERIVATION OF EQ.(9) IN THE MAIN TEXT

After the interaction, the detector will be brought to the following final state

$$\rho_D = \sum_{a=0}^{1} p_a \varrho_a = \varrho_+ + z\varrho_- \tag{13}$$

with $p_a = \frac{1\pm z}{2}$ and $2\varrho_\pm = \varrho_0 \pm \varrho_1$. For convenience we denote by $P_\pm$ the projector to the non-negative or negative, respectively, eigenspace of $p_0\varrho_0 - p_1\varrho_1$. The distinguishability now becomes

$$\mathcal{D} = \mathrm{Tr}[(P_+ - P_-)(z\varrho_+ + \varrho_-)] = \delta_- + z\delta_+ \tag{14}$$

with $\delta_\pm = \mathrm{Tr}(P_+ - P_-)\varrho_\pm$. Since the quantum Fisher information is the largest Fisher information over all possible measurements, by performing a special two-outcome measurement $\{P_+, P_-\}$, we have

$$F_z[\rho_D] \ge \sum_\pm \frac{(\mathrm{Tr}P_\pm \dot{\rho}_D)^2}{\mathrm{Tr}P_\pm \rho_D} = \frac{\delta_-^2}{1-(\delta_+ + z\delta_-)^2}$$

leading to, using Eq.(14),

$$\begin{aligned} &(1-z^2)^2 F_z[\rho_D] + z^2 - \mathcal{D}^2 \\ \ge\ &\frac{(1-z^2)^2\delta_-^2 + (z-\mathcal{D}^2)[1-(\delta_+ + z\delta_-)^2]}{1-(\delta_+ + z\delta_-)^2} \\ =\ &\frac{[(1+z^2)\delta_+\delta_- - z(1-\delta_+^2-\delta_-^2)]^2}{1-(\delta_+ + z\delta_-)^2} \ge 0. \end{aligned}$$

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[1] S. L. Braunstein and C.M. Caves 1994 Phys. Rev. Lett. 72, 3439