Supplementary Materials: Superconductivity near the $(2+1)$-Dimensional Ferromagnetic Quantum Critical Point

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I. DERIVATION OF THE BCS SELF-CONSISTENT EQUATION

Here we provide a detailed derivation of Eq. (5) in the main text. First, we can decouple the order parameter into its expectation value $\chi_\sigma = \langle c_{i2\sigma} c_{i1\sigma}\rangle$(due to the transnational symmetry, the expectation value does not depend on the site label i) and fluctuation. Then we only keep the attractive interaction terms in Eq. (5) in the main text up to the first order of the fluctuation

$$c^\dagger_{i1\sigma} c^\dagger_{i2\sigma} c_{i2\sigma} c_{i1\sigma} = -|\chi_\uparrow|^2 + \chi_\sigma c^\dagger_{i1\sigma} c^\dagger_{i2\sigma} + \chi^*_\sigma c_{i2\sigma} c_{i1\sigma} \tag{1}$$

Combing the H_f and H_{eff} together and rewriting them into momentum space, we have

$$\begin{aligned} H_{\text{MF}} = \sum_{\boldsymbol{k}\in\frac{1}{2}\text{BZ}} [\epsilon(\boldsymbol{k}) \Big(& c^\dagger_{\boldsymbol{k}1\uparrow} c_{\boldsymbol{k}1\uparrow} + c^\dagger_{\boldsymbol{k}1\downarrow} c_{\boldsymbol{k}1\downarrow} + c^\dagger_{\boldsymbol{k}2\uparrow} c_{\boldsymbol{k}2\uparrow} + c^\dagger_{\boldsymbol{k}2\downarrow} c_{\boldsymbol{k}2\downarrow} + c^\dagger_{-\boldsymbol{k}1\uparrow} c_{-\boldsymbol{k}1\uparrow} + c^\dagger_{-\boldsymbol{k}1\downarrow} c_{-\boldsymbol{k}1\downarrow} + c^\dagger_{-\boldsymbol{k}2\uparrow} c_{-\boldsymbol{k}2\uparrow} + c^\dagger_{-\boldsymbol{k}2\downarrow} c_{-\boldsymbol{k}2\downarrow} \Big) \\ & - \frac{\lambda^2 \chi_\uparrow}{8\alpha(h-h_c)} (c^\dagger_{\boldsymbol{k}1\uparrow} c^\dagger_{-\boldsymbol{k}2\uparrow} + c^\dagger_{-\boldsymbol{k}1\uparrow} c^\dagger_{\boldsymbol{k}2\uparrow}) - \frac{\lambda^2 \chi_\downarrow}{8\alpha(h-h_c)} (c^\dagger_{\boldsymbol{k}1\downarrow} c^\dagger_{-\boldsymbol{k}2\downarrow} + c^\dagger_{-\boldsymbol{k}1\downarrow} c_{\boldsymbol{k}2\downarrow}) \\ & - \frac{\lambda^2 \chi^*_\uparrow}{8\alpha(h-h_c)} (c_{\boldsymbol{k}2\uparrow} c_{-\boldsymbol{k}1\uparrow} + c_{-\boldsymbol{k}2\uparrow} c_{\boldsymbol{k}1\uparrow}) - \frac{\lambda^2 \chi^*_\downarrow}{8\alpha(h-h_c)} (c_{\boldsymbol{k}2\downarrow} c_{-\boldsymbol{k}1\downarrow} + c_{-\boldsymbol{k}2\downarrow} c_{\boldsymbol{k}1\downarrow})] \end{aligned} \tag{2}$$

where $\epsilon(\boldsymbol{k}) = -t(2\cos k_x + 2\cos k_y) - \mu$ is the energy dispersion for the fermions and the momentum summation is restricted to one half of the Brillouin zone to avoid double counting. We also drop off the constant term in the final expression. Next, we can write the equation into matrix form as

$$H_{\text{MF}} = \sum_{\boldsymbol{k}\in\frac{1}{2}\text{BZ}} \begin{pmatrix} c^\dagger_{\boldsymbol{k}1\uparrow} & c^\dagger_{\boldsymbol{k}1\uparrow} & c^\dagger_{\boldsymbol{k}1\downarrow} & c^\dagger_{\boldsymbol{k}2\uparrow} & c_{-\boldsymbol{k}1\uparrow} & c_{-\boldsymbol{k}1\downarrow} & c_{-\boldsymbol{k}2\uparrow} & c_{-\boldsymbol{k}2\downarrow} \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} \epsilon(\boldsymbol{k}) & 0 & 0 & 0 & 0 & 0 & -\tilde{\chi}_\uparrow & 0 \\ 0 & \epsilon(\boldsymbol{k}) & 0 & 0 & 0 & 0 & 0 & -\tilde{\chi}_\downarrow \\ 0 & 0 & \epsilon(\boldsymbol{k}) & 0 & \tilde{\chi}_\uparrow & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon(\boldsymbol{k}) & 0 & \tilde{\chi}_\downarrow & 0 & 0 \\ 0 & 0 & \tilde{\chi}^*_\uparrow & 0 & -\epsilon(\boldsymbol{k}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\chi}^*_\downarrow & 0 & -\epsilon(\boldsymbol{k}) & 0 & 0 \\ -\tilde{\chi}^*_\uparrow & 0 & 0 & 0 & 0 & 0 & -\epsilon(\boldsymbol{k}) & 0 \\ 0 & -\tilde{\chi}^*_\downarrow & 0 & 0 & 0 & 0 & 0 & -\epsilon(\boldsymbol{k}) \end{pmatrix} \begin{pmatrix} c_{\boldsymbol{k}1\uparrow} \\ c_{\boldsymbol{k}1\downarrow} \\ c_{\boldsymbol{k}2\uparrow} \\ c_{\boldsymbol{k}2\downarrow} \\ c^\dagger_{-\boldsymbol{k}1\uparrow} \\ c^\dagger_{-\boldsymbol{k}1\downarrow} \\ c^\dagger_{-\boldsymbol{k}2\uparrow} \\ c^\dagger_{-\boldsymbol{k}2\downarrow} \end{pmatrix} \tag{4}$$

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where $\tilde{\chi}_\sigma = \frac{\lambda^2 \chi_\sigma}{8\alpha(h-h_c)}$. Then, we can use the unitary transformation to diagonalize the mean-field hamiltionian matrix $\mathcal{H}$ as

$$\mathcal{D} = \mathcal{U}^\dagger \mathcal{H} \mathcal{U} \tag{5}$$

by

$$\mathcal{U} = \begin{pmatrix}
\sqrt{\frac{\sqrt{B}-a}{2\sqrt{B}}}\frac{b}{|b|} & 0 & -\sqrt{\frac{\sqrt{B}+a}{2\sqrt{B}}}\frac{b}{|b|} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\frac{\sqrt{C}-a}{2\sqrt{C}}}\frac{c}{|c|} & 0 & -\sqrt{\frac{\sqrt{C}+a}{2\sqrt{C}}}\frac{c}{|c|} & 0 \\
0 & -\sqrt{\frac{\sqrt{B}-a}{2\sqrt{B}}}\frac{b}{|b|} & 0 & \sqrt{\frac{\sqrt{B}+a}{2\sqrt{B}}}\frac{b}{|b|} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{\sqrt{C}-a}{2\sqrt{C}}}\frac{c}{|c|} & 0 & -\sqrt{\frac{\sqrt{C}+a}{2\sqrt{C}}}\frac{c}{|c|} \\
0 & \frac{|b|}{\sqrt{2}\sqrt{B-a\sqrt{B}}} & 0 & \frac{|b|}{\sqrt{2}\sqrt{B+a\sqrt{B}}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{|c|}{\sqrt{2}\sqrt{C-a\sqrt{C}}} & 0 & \frac{|c|}{\sqrt{2}\sqrt{C+a\sqrt{C}}} \\
\frac{|b|}{\sqrt{2}\sqrt{B-a\sqrt{B}}} & 0 & \frac{|b|}{\sqrt{2}\sqrt{B+a\sqrt{B}}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{|c|}{\sqrt{2}\sqrt{C-a\sqrt{C}}} & 0 & \frac{|c|}{\sqrt{2}\sqrt{C+a\sqrt{C}}} & 0
\end{pmatrix} \tag{6}$$

$$\mathcal{D} = \begin{pmatrix}
-\sqrt{B} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\sqrt{B} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{B} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{B} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{C} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{C} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{C} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{C}
\end{pmatrix} \tag{7}$$

where $a = \epsilon(\boldsymbol{k})$, $b = \tilde{\chi}_\uparrow$, $c = \tilde{\chi}_\downarrow$, $B = \epsilon(\boldsymbol{k})^2 + |\tilde{\chi}_\uparrow|^2$ and $C = \epsilon(\boldsymbol{k})^2 + |\tilde{\chi}_\downarrow|^2$.

Hence we can define the Bogoliubov quasi-particles as following

$$\begin{pmatrix} \gamma_{1\boldsymbol{k}} \\ \gamma_{2\boldsymbol{k}} \\ \gamma_{3\boldsymbol{k}} \\ \gamma_{4\boldsymbol{k}} \\ \gamma_{5\boldsymbol{k}} \\ \gamma_{6\boldsymbol{k}} \\ \gamma_{7\boldsymbol{k}} \\ \gamma_{8\boldsymbol{k}} \end{pmatrix} = \mathcal{U}^\dagger \begin{pmatrix} c_{\boldsymbol{k}1\uparrow} \\ c_{\boldsymbol{k}1\downarrow} \\ c_{\boldsymbol{k}2\uparrow} \\ c_{\boldsymbol{k}2\downarrow} \\ c^\dagger_{-\boldsymbol{k}1\uparrow} \\ c^\dagger_{-\boldsymbol{k}1\downarrow} \\ c^\dagger_{-\boldsymbol{k}2\uparrow} \\ c^\dagger_{-\boldsymbol{k}2\downarrow} \end{pmatrix} \tag{8}$$

and obtain the gap equation by rewriting the expectation value of the order parameter χ_σ by Bogoliubov quasi-particles

$$\begin{aligned}
\chi_\uparrow &= \langle c_{i2\uparrow} c_{i1\uparrow} \rangle \\
&= \frac{1}{L^2} \sum_{\boldsymbol{k}} \langle c_{\boldsymbol{k}2\uparrow} c_{-\boldsymbol{k}1\uparrow} \rangle \\
&= \frac{1}{L^2} \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} (\langle c_{\boldsymbol{k}2\uparrow} c_{-\boldsymbol{k}1\uparrow} \rangle + \langle c_{-\boldsymbol{k}2\uparrow} c_{\boldsymbol{k}1\uparrow} \rangle) \\
&= \frac{1}{L^2} \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} [\langle (-\sqrt{\frac{\sqrt{B}-a}{2\sqrt{B}}} \frac{b}{|b|} \gamma_{2\boldsymbol{k}} + \sqrt{\frac{\sqrt{B}+a}{2\sqrt{B}}} \frac{b}{|b|} \gamma_{4\boldsymbol{k}}) (\frac{|b|}{\sqrt{2}\sqrt{B-a\sqrt{B}}} \gamma_{2\boldsymbol{k}}^\dagger + \frac{|b|}{\sqrt{2}\sqrt{B+a\sqrt{B}}} \gamma_{4\boldsymbol{k}}^\dagger) \rangle \\
&\quad + \langle (\frac{|b|}{\sqrt{2}\sqrt{B-a\sqrt{B}}} \gamma_{1\boldsymbol{k}}^\dagger + \frac{|b|}{\sqrt{2}\sqrt{B+a\sqrt{B}}} \gamma_{3\boldsymbol{k}}^\dagger) (\sqrt{\frac{\sqrt{B}-a}{2\sqrt{B}}} \frac{b}{|b|} \gamma_{1\boldsymbol{k}} - \sqrt{\frac{\sqrt{B}+a}{2\sqrt{B}}} \frac{b}{|b|} \gamma_{3\boldsymbol{k}}) \rangle] \\
&= \frac{1}{L^2} \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \frac{b}{2\sqrt{B}} (-\langle \gamma_{2\boldsymbol{k}} \gamma_{2\boldsymbol{k}}^\dagger \rangle + \langle \gamma_{4\boldsymbol{k}} \gamma_{4\boldsymbol{k}}^\dagger \rangle + \langle \gamma_{1\boldsymbol{k}}^\dagger \gamma_{1\boldsymbol{k}} \rangle - \langle \gamma_{3\boldsymbol{k}}^\dagger \gamma_{3\boldsymbol{k}} \rangle) \\
&= \frac{\lambda^2 \chi_\uparrow}{8\alpha(h-h_c)} \frac{1}{L^2} \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \frac{1}{\sqrt{\epsilon(\boldsymbol{k})^2 + |\tilde{\chi}_\uparrow|^2}} \frac{\exp(\frac{\sqrt{\epsilon(\boldsymbol{k})^2+|\tilde{\chi}_\uparrow|^2}}{T}) - \exp(-\frac{\sqrt{\epsilon(\boldsymbol{k})^2+|\tilde{\chi}_\uparrow|^2}}{T})}{\exp(\frac{(\sqrt{\epsilon(\boldsymbol{k})^2+|\tilde{\chi}_\uparrow|^2}}{T}) + \exp(-\frac{\sqrt{\epsilon(\boldsymbol{k})^2+|\tilde{\chi}_\uparrow|^2}}{T}) + 2}
\end{aligned} \tag{9}$$

With further simplification, we finally get the Eq. (5) in the main text:

$$1 = \frac{\lambda^2}{8\alpha(h-h_c)} \frac{1}{L^2} \sum_{\boldsymbol{k}\in\frac{1}{2}\mathrm{BZ}} \frac{\tanh(\sqrt{\epsilon(\boldsymbol{k})^2 + |\Delta_\sigma|^2}/(2T))}{\sqrt{\epsilon(\boldsymbol{k})^2 + |\Delta_\sigma|^2}} \tag{10}$$

where we rewrite $\Delta_\sigma = \tilde{\chi}_\sigma = \frac{\lambda^2}{8\alpha(h-h_c)L^2} \sum_{\boldsymbol{k}} \langle c_{\boldsymbol{k}2\sigma} c_{-\boldsymbol{k}1\sigma} \rangle$