Supplementary Material: Vector Spatiotemporal Solitons and Their Memory Features in Cold Rydberg Gases

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Text A: Expansion equations of optical Bloch equation

The density equations ρ of the system is obtained by substituting the Hamiltonian to the optical Bloch equation , $\partial\rho/\partial t=-i\left[\hat{H},\rho\right]/\hbar-\Gamma[\rho]$. For the 5×5 density matrix, we get the following expansion equations of optical Bloch equation.

$$i(\frac{\partial}{\partial t}+\Gamma_{21}+\Gamma_{31})\rho_{11}-i(\Gamma_{12}\rho_{22}+\Gamma_{13}\rho_{33}+\Gamma_{14}\rho_{44})-\Omega_p\rho_{41}{}^{*}+\Omega_p^{*}\rho_{41}=0, \quad \text{(A1)}$$

$$i(\frac{\partial}{\partial t}+\Gamma_{12})\rho_{22}-i(\Gamma_{21}\rho_{11}+\Gamma_{24}\rho_{44})-\Omega_s\rho_{42}{}^{*}+\Omega_s^{*}\rho_{42}=0, \quad \text{(A2)}$$

$$i(\frac{\partial}{\partial t}+\Gamma_{13})\rho_{33}-i(\Gamma_{31}\rho_{11}+\Gamma_{34}\rho_{44})-\Omega_c\rho_{43}{}^{*}+\Omega_c^{*}\rho_{43}=0, \quad \text{(A3)}$$

$$i(\frac{\partial}{\partial t}+\Gamma_{14}+\Gamma_{24}+\Gamma_{34})\rho_{44}-i\Gamma_{45}\rho_{55}+\Omega_p\rho_{41}^{*}-\Omega_p^{*}\rho_{41}+\Omega_s\rho_{42}^{*}-\Omega_s^{*}\rho_{42}+\Omega_c\rho_{43}^{*}-\Omega_c^{*}\rho_{43}+\Omega_a^{*}\rho_{54}-\Omega_a\rho_{54}^{*}=0, \quad \text{(A4)}$$

$$i(\frac{\partial}{\partial t}+\Gamma_{45})\rho_{55}-\Omega_a^{*}\rho_{54}+\Omega_a\rho_{45}=0, \quad \text{(A5)}$$

$$(i\frac{\partial}{\partial t}+d_{21})\rho_{21}+\Omega_s^{*}\rho_{41}-\Omega_p\rho_{42}^{*}=0, \quad \text{(A6)}$$

$$(i\frac{\partial}{\partial t}+d_{31})\rho_{31}+\Omega_c^{*}\rho_{41}-\Omega_p\rho_{43}^{*}=0, \quad \text{(A7)}$$

$$(i\frac{\partial}{\partial t}+d_{41})\rho_{41}+\Omega_p(\rho_{11}-\rho_{44})+\Omega_s\rho_{21}+\Omega_c\rho_{31}+\Omega_a^{*}\rho_{51}=0, \quad \text{(A8)}$$

$$(i\frac{\partial}{\partial t}+d_{51})\rho_{51}+\Omega_a\rho_{41}-\Omega_p\rho_{54}-N_a\int d^3\mathbf{r}'V(\mathbf{r}'-\mathbf{r})\rho_{55,51}(\mathbf{r}',\mathbf{r},t)=0, \quad \text{(A9)}$$

$$(i\frac{\partial}{\partial t}+d_{32})\rho_{32}+\Omega_c^{*}\rho_{42}-\Omega_s\rho_{43}^{*}=0, \quad \text{(A10)}$$

$$(i\frac{\partial}{\partial t}+d_{42})\rho_{42}+\Omega_s(\rho_{22}-\rho_{44})+\Omega_p\rho_{21}^{*}+\Omega_c\rho_{32}+\Omega_a^{*}\rho_{52}=0, \quad \text{(A11)}$$

$$(i\frac{\partial}{\partial t}+d_{52})\rho_{52}+\Omega_a\rho_{42}-\Omega_s\rho_{54}-N_a\int d^3\mathbf{r}'V(\mathbf{r}'-\mathbf{r})\rho_{55,52}(\mathbf{r}',\mathbf{r},t)=0, \quad \text{(A12)}$$

$$(i\frac{\partial}{\partial t}+d_{43})\rho_{43}+\Omega_c(\rho_{33}-\rho_{44})+\Omega_s\rho_{32}^*+\Omega_p\rho_{31}^*+\Omega_a^*\rho_{53}=0, \quad \text{(A13)}$$

$$(i\frac{\partial}{\partial t}+d_{53})\rho_{53}+\Omega_a\rho_{43}-\Omega_c\rho_{54}-N_a\int d^3\mathbf{r}'V(\mathbf{r}'-\mathbf{r})\rho_{55,53}(\mathbf{r}',\mathbf{r},t)=0, \quad \text{(A14)}$$

$$(i\frac{\partial}{\partial t}+d_{54})\rho_{54}+\Omega_a(\rho_{44}-\rho_{55})-\Omega_p^*\rho_{51}-\Omega_s^*\rho_{52}-\Omega_c^*\rho_{53}-N_a\int d^3\mathbf{r}'V(\mathbf{r}'-\mathbf{r})\rho_{55,54}(\mathbf{r}',\mathbf{r},t)=0, \quad \text{(A15)}$$

where $d_{jl}=\Delta_j-\Delta_l+i\gamma_{jl}$ ($i,j=1,2,3,4;i\neq j$), and $\gamma_{jl}=(\Gamma_j+\Gamma_l)/2+\gamma_{ij}^{dep}$ with $\Gamma_l=\sum_{j<l}\Gamma_{jl}$. Here Γ_{jl} denotes the spontaneous emission decay rate between states $|j\rangle$ and $|l\rangle$, and γ_{ij}^{dep} is the dephasing rate between the states $|j\rangle$ and $|l\rangle$.

Text B: Expansion equations of zeroth order density matrix elements and the first order solutions

(i) At the zeroth ($m=0$) order, equations for $\rho_{32}^{(0)},\rho_{42}^{(0)},\rho_{43}^{(0)},\rho_{52}^{(0)},\rho_{53}^{(0)}$, and $\rho_{54}^{(0)}$ are given by

$$\begin{bmatrix} d_{32} & \Omega_c^* & 0 & 0 & 0 & 0 \\ \Omega_c & d_{42} & 0 & \Omega_a^* & 0 & 0 \\ 0 & 0 & d_{43} & 0 & \Omega_a^* & 0 \\ 0 & \Omega_a & 0 & d_{52} & 0 & 0 \\ 0 & 0 & \Omega_a & 0 & d_{53} & 0 \\ 0 & 0 & 0 & 0 & -\Omega_c^* & d_{54} \end{bmatrix}\begin{bmatrix} \rho_{32}^{(0)} \\ \rho_{42}^{(0)} \\ \rho_{43}^{(0)} \\ \rho_{52}^{(0)} \\ \rho_{53}^{(0)} \\ \rho_{54}^{(0)} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \Omega_c(\rho_{44}^{(0)}-\rho_{33}^{(0)}) \\ 0 \\ 0 \\ -\Omega_a\rho_{44}^{(0)} \end{bmatrix}, \quad \text{(B1)}$$

Equations for $\rho_{11}^{(0)},\rho_{22}^{(0)},\rho_{33}^{(0)}$, and $\rho_{44}^{(0)}$ read

$$\begin{bmatrix} -(\Gamma_{21}+\Gamma_{31}) & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & -\Gamma_{12} & 0 & \Gamma_{24} \\ \Gamma_{31} & 0 & -\Gamma_{13} & \Gamma_{34} \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \rho_{11}^{(0)} \\ \rho_{22}^{(0)} \\ \rho_{33}^{(0)} \\ \rho_{44}^{(0)} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ i(\Omega_c\rho_{43}^{*(0)}-\Omega_c^*\rho_{43}^{(0)}) \\ 1 \end{bmatrix}, \quad \text{(B2)}$$

And $\rho_{31}^{(0)}=\rho_{21}^{(0)}=\rho_{41}^{(0)}=\rho_{51}^{(0)}=0,\rho_{32}^{(0)}=\rho_{42}^{(0)}=\rho_{52}^{(0)}=\rho_{55}^{(0)}=0$.

(ii) At the first ($m=1$) order, the solution for nonzero matrix elements reads

$$\rho_{31}^{(1)}=-\frac{\Omega_c^*(\omega+d_{21})(\omega+d_{31})(\rho_{11}^{(0)}-\rho_{44}^{(0)})+|\Omega_c|^2\rho_{43}^{*(0)}}{D_1}F_1e^{i\theta_p}=a_{31}^{(1)}F_1e^{i\theta_p} \quad \text{(B3)}$$

$$\rho_{41}^{(1)}=\frac{(\omega+d_{21})(\omega+d_{31})(\rho_{11}^{(0)}-\rho_{44}^{(0)})+\Omega_c\rho_{43}^{*(0)}}{D_1}F_1e^{i\theta_p}=a_{41}^{(1)}F_1e^{i\theta_p} \quad \text{(B4)}$$

$$\rho_{51}^{(1)}=-\frac{\Omega_a(\omega+d_{21})(\omega+d_{31})(\rho_{11}^{(0)}-\rho_{44}^{(0)})+\Omega_a\Omega_c\rho_{43}^{*(0)}}{D_1}F_1e^{i\theta_p}=a_{51}^{(1)}F_1e^{i\theta_p} \quad \text{(B5)}$$

$$\rho_{32}^{(1)}=-\frac{\Omega_c^*(\omega+d_{32})(\omega+d_{52})(\rho_{22}^{(0)}-\rho_{44}^{(0)})+(\omega+d_{32})\Omega_a^*\Omega_c^*\rho_{54}^{(0)}+(\omega+d_{52})|\Omega_c|^2\rho_{43}^{*(0)}}{D_2}F_2e^{i\theta_s}=a_{32}^{(1)}F_2e^{i\theta_s} \quad \text{(B6)}$$

$$\rho_{42}^{(1)}=\frac{(\omega+d_{32})(\omega+d_{52})(\rho_{22}^{(0)}-\rho_{44}^{(0)})+(\omega+d_{32})\Omega_a^*\rho_{54}^{(0)}+(\omega+d_{52})\Omega_c\rho_{43}^{*(0)}}{D_2(\omega+d_{32})}F_2e^{i\theta_s}=a_{42}^{(1)}F_2e^{i\theta_s},\quad(B7)$$

$$\rho_{52}^{(1)}=-\frac{\Omega_a(\omega+d_{32})(\omega+d_{52})(\rho_{22}^{(0)}-\rho_{44}^{(0)})+(\omega+d_{32})|\Omega_a|^2\rho_{54}^{(0)}+(\omega+d_{52})\Omega_c\Omega_a\rho_{43}^{*(0)}}{D_2(\omega+d_{52})}F_2e^{i\theta_s}=a_{52}^{(1)}F_2e^{i\theta_s},\quad(B8)$$

with other $\rho_{jl}^{(1)}=0$.

Text C: The second order solution

At the second order, the solution for nonzero matrix elements reads

$$\rho_{21}^{(2)}=\frac{1}{\omega+d_{21}}(\rho_{42}^{*(1)}F_1e^{i\theta_p}-\rho_{41}^{(1)}F_2^*e^{-i\theta_s^*}),\quad(C1)$$

$$\rho_{31}^{(2)}=\frac{-1}{\omega+d_{31}}(\Omega_c^*\rho_{41}^{(2)}+ia_{31}^{(1)})\frac{\partial}{\partial t_1}F_1e^{i\theta_p},\quad(C2)$$

$$\rho_{41}^{(2)}=\frac{i\left[(\omega+d_{31})(\omega+d_{51})a_{41}^{(1)}+(\omega+d_{51})\Omega_ca_{31}^{(1)}-(\omega+d_{31})\Omega_a^*\rho_{51}^{(1)}\right]}{D_1}\frac{\partial}{\partial t_1}F_1e^{i\theta_p},\quad(C3)$$

$$\rho_{51}^{(2)}=\frac{-1}{\omega+d_{31}}(\Omega_a a_{41}^{(2)}+ia_{51}^{(1)})\frac{\partial}{\partial t_1}F_1e^{i\theta_p},\quad(C4)$$

$$\rho_{32}^{(2)}=\frac{-1}{\omega+d_{32}}(\Omega_c^*\rho_{42}^{(2)}+ia_{32}^{(1)})\frac{\partial}{\partial t_1}F_2e^{i\theta_s},\quad(C5)$$

$$\rho_{42}^{(2)}=\frac{i\left[(\omega+d_{32})(\omega+d_{52})a_{42}^{(1)}-(\omega+d_{52})\Omega_ca_{32}^{(1)}-(\omega+d_{32})\Omega_a^*\rho_{52}^{(1)}\right]}{D_2}\frac{\partial}{\partial t_1}F_2e^{i\theta_s},\quad(C6)$$

$$\rho_{52}^{(2)}=\frac{-1}{\omega+d_{52}}(\Omega_a a_{42}^{(2)}+ia_{52}^{(1)})\frac{\partial}{\partial t_1}F_2e^{i\theta_s},\quad(C7)$$

$$\rho_{11}^{(2)}=a_{111}^{(2)}|F_1|^2e^{(-2\bar{\alpha}_1z_2)}+a_{112}^{(2)}|F_2|^2e^{(-2\bar{\alpha}_2z_2)},\quad(C8)$$

$$\rho_{22}^{(2)}=a_{221}^{(2)}|F_1|^2e^{(-2\bar{\alpha}_1z_2)}+a_{222}^{(2)}|F_2|^2e^{(-2\bar{\alpha}_2z_2)},\quad(C9)$$

$$\rho_{33}^{(2)}=a_{331}^{(2)}|F_1|^2e^{(-2\bar{\alpha}_1z_2)}+a_{332}^{(2)}|F_2|^2e^{(-2\bar{\alpha}_2z_2)},\quad(C10)$$

$$\rho_{44}^{(2)}=a_{441}^{(2)}|F_1|^2e^{(-2\bar{\alpha}_1z_2)}+a_{442}^{(2)}|F_2|^2e^{(-2\bar{\alpha}_2z_2)},\quad(C11)$$

$$\rho_{43}^{(2)}=a_{431}^{(2)}|F_1|^2e^{(-2\bar{\alpha}_1z_2)}+a_{432}^{(2)}|F_2|^2e^{(-2\bar{\alpha}_2z_2)},\quad(C12)$$

$$\rho_{53}^{(2)}=a_{531}^{(2)}|F_1|^2e^{(-2\bar{\alpha}_1z_2)}+a_{532}^{(2)}|F_2|^2e^{(-2\bar{\alpha}_2z_2)},\quad(C13)$$

$$\rho_{54}^{(2)}=a_{541}^{(2)}|F_1|^2e^{(-2\bar{\alpha}_1z_2)}+a_{542}^{(2)}|F_2|^2e^{(-2\bar{\alpha}_2z_2)},\quad(C14)$$

with

$$a_{111}^{(2)}=-\frac{(2A_1+2C_{11}+H_{11})X_1-(A_1+C_{11})\Gamma_{12}\Gamma_{13}\Gamma_{45}+C_{11}X_3-A_1(\Gamma_{13}\Gamma_{24}+\Gamma_{12}\Gamma_{34})\Gamma_{45}}{X_2}$$

$$a_{112}^{(2)} = -\frac{(2B_1 + 2C_{12} + H_{12})X_1 - (B_1 + C_{12})\Gamma_{12}\Gamma_{13}\Gamma_{45} + C_{12}X_3 + B_1X_4}{X_2}$$

$$a_{221}^{(2)} = -\frac{(2A_1 + 2C_{11} + H_{11})Y_1 + A_1X_5 + C_{11}X_6}{X_2}$$

$$a_{222}^{(2)} = -\frac{(2B_1 + 2C_{12} + H_{12})Y_1 - B_1[Y_2 + (\Gamma_{21}\Gamma_{34} + \Gamma_{13}\Gamma_{21})\Gamma_{45}] + C_{12}X_6}{X_2}$$

$$a_{331}^{(2)} = -\frac{(2A_1 + 2C_{11} + H_{11})Z_1 - C_{11}[Z_2 + (\Gamma_{12} + \Gamma_{24})\Gamma_{31}\Gamma_{45}] + A_1X_7}{X_3}$$

$$a_{332}^{(2)} = -\frac{(2B_1 + 2C_{12} + H_{12})Z_1 - C_{12}[Z_2 + (\Gamma_{12} + \Gamma_{24})\Gamma_{31}\Gamma_{45}] + B_1(Z_3 - \Gamma_{12}\Gamma_{31}\Gamma_{45})}{X_3}$$

$$a_{431}^{(2)} = \frac{(\omega + d_{53})\left[|\Omega_a|^2\,\Omega_c(a_{111}^{(2)} - a_{221}^{(2)} - a_{331}^{(2)} - 2a_{441}^{(2)}) - D_4\Omega_c(a_{331}^{(2)} - a_{441}^{(2)}) - \Omega_a\Omega_c^* a_{51}^{(1)} - D_4 a_{31}^{*(1)}\right]}{D_3D_4 - |\Omega_a|^2\,|\Omega_c|^2}$$

$$a_{432}^{(2)} = \frac{(\omega + d_{53})\left[|\Omega_a|^2\,\Omega_c(a_{112}^{(2)} - a_{222}^{(2)} - a_{332}^{(2)} - 2a_{442}^{(2)}) - D_4\Omega_c(a_{332}^{(2)} - a_{442}^{(2)}) - \Omega_a\Omega_c^* a_{52}^{(1)} - D_4 a_{32}^{*(1)}\right]}{D_3D_4 - |\Omega_a|^2\,|\Omega_c|^2}$$

$$a_{531}^{(2)} = \frac{\Omega_a\left[|\Omega_a|^2\,\Omega_c(a_{111}^{(2)} - a_{221}^{(2)} - a_{331}^{(2)} - 2a_{441}^{(2)}) - D_4\Omega_c(a_{331}^{(2)} - a_{441}^{(2)}) - \Omega_a\Omega_c^* a_{51}^{(1)} - D_4 a_{31}^{*(1)}\right]}{D_3D_4 - |\Omega_a|^2\,|\Omega_c|^2}$$

$$-\frac{\left[\Omega_c a_{31}^{*(1)} - [\Omega_c]^2(a_{331}^{(2)} - a_{441}^{(2)})\right]}{\Omega_c\Omega_a^*} - \frac{D_4 a_{431}^{(2)}}{\Omega_a^*(\omega + d_{53})}$$

$$a_{532}^{(2)} = \frac{\Omega_a\left[|\Omega_a|^2\,\Omega_c(a_{112}^{(2)} - a_{222}^{(2)} - a_{332}^{(2)} - 2a_{442}^{(2)}) - D_4\Omega_c(a_{332}^{(2)} - a_{442}^{(2)}) - \Omega_a\Omega_c^* a_{52}^{(1)} - D_4 a_{32}^{*(1)}\right]}{D_3D_4 - |\Omega_a|^2\,|\Omega_c|^2}$$

$$+\frac{\left[\Omega_c a_{32}^{*(1)} - [\Omega_c]^2(a_{332}^{(2)} - a_{442}^{(2)})\right]}{\Omega_c\Omega_a^*} - \frac{D_4 a_{432}^{(2)}}{\Omega_a^*(\omega + d_{53})}$$

$$a_{541}^{(2)} = -\frac{(\omega + d_{53})\left[\Omega_c(a_{331}^{(2)} - a_{441}^{(2)}) + a_{31}^{*(1)}\right] + D_4 a_{431}^{(2)}}{\Omega_c\Omega_a^*}$$

$$a_{542}^{(2)} = \frac{(\omega + d_{53})\left[a_{32}^{*(1)} - \Omega_c(a_{332}^{(2)} - a_{442}^{(2)})\right] - D_4 a_{432}^{(2)}}{\Omega_c\Omega_a^*}$$

where

$$A_1 = \frac{\Omega_c\rho_{43}^{*(0)} - \Omega_c^*\rho_{43}^{(0)}}{D_1}$$

$$B_1 = \frac{(\omega + d_{32})\left(\Omega_a^*\rho_{54}^{(0)} - \Omega_a\rho_{54}^{*(0)}\right) + (\omega + d_{52})\left(\Omega_c\rho_{43}^{*(0)} - \Omega_c^*\rho_{43}^{(0)}\right)}{D_2}$$

$$C_{11} = \frac{(\omega + d_{53})\left[[\Omega_c]^2\,\Omega_a^* a_{51}^{*(1)} - \Omega_a\left[\Omega_c^*\right]^2 a_{51}^{(1)} + \Omega_c D_4 a_{31}^{(1)} - \Omega_c^* D_4 a_{31}^{*(1)}\right]}{D_3D_4 - |\Omega_a|^2\,|\Omega_c|^2}$$

$$C_{12}=\frac{(\omega+d_{53})\left[\left[\Omega_c\right]^2\Omega_a^*a_{52}^{*(1)}-\Omega_a\left[\Omega_c^*\right]^2a_{52}^{(1)}+\Omega_cD_4a_{32}^{(1)}-\Omega_c^*D_4a_{32}^{*(1)}\right]}{D_3D_4-\left|\Omega_a\right|^2\left|\Omega_c\right|^2}$$

$$D_3=(\omega+d_{53})(\omega+d_{43})-\left|\Omega_a\right|^2$$

$$D_4=(\omega+d_{53})(\omega+d_{54})-\left|\Omega_c\right|^2$$

$$H_{11}=\frac{D_4(\omega+d_{54})}{D_3D_4-\left|\Omega_a\right|^2\left|\Omega_c\right|^2}\left[\frac{\Omega_c^*\Omega_a a_{51}^{(1)}+D_4a_{31}^{*(1)}}{\Omega_c}-\frac{\Omega_a^*\Omega_c a_{51}^{*(1)}+D_4a_{31}^{(1)}}{\Omega_c^*}\right]+(\omega+d_{53})(\frac{a_{31}^{(1)}}{\Omega_c^*}-\frac{a_{31}^{*(1)}}{\Omega_c})$$

$$H_{12}=\frac{D_4(\omega+d_{54})}{D_3D_4-\left|\Omega_a\right|^2\left|\Omega_c\right|^2}\left[\frac{\Omega_c^*\Omega_a a_{52}^{(1)}+D_4a_{32}^{*(1)}}{\Omega_c}-\frac{\Omega_a^*\Omega_c a_{52}^{*(1)}+D_4a_{32}^{(1)}}{\Omega_c^*}\right]+(\omega+d_{53})(\frac{a_{32}^{*(1)}}{\Omega_c}-\frac{a_{32}^{(1)}}{\Omega_c^*})$$

$$X_1=\Gamma_{12}\Gamma_{13}(\Gamma_{14}+\Gamma_{24}+\Gamma_{34})$$

$$X_2=(\Gamma_{12}\Gamma_{13}+\Gamma_{21}\Gamma_{13}+\Gamma_{12}\Gamma_{31})(\Gamma_{14}+\Gamma_{24}+\Gamma_{34})\Gamma_{45}$$

$$X_3=(\Gamma_{12}\Gamma_{14}+\Gamma_{12}\Gamma_{24}-\Gamma_{13}\Gamma_{24})\Gamma_{45}$$

$$X_4=(\Gamma_{13}\Gamma_{14}+\Gamma_{13}\Gamma_{34}-\Gamma_{12}\Gamma_{34})\Gamma_{45}$$

$$X_5=(\Gamma_{13}\Gamma_{24}+\Gamma_{24}\Gamma_{31}-\Gamma_{13}\Gamma_{21}-\Gamma_{34}\Gamma_{21})\Gamma_{45}$$

$$X_6=(\Gamma_{13}\Gamma_{24}+\Gamma_{24}\Gamma_{31}+\Gamma_{21}\Gamma_{24}-\Gamma_{13}\Gamma_{21}-\Gamma_{14}\Gamma_{21})\Gamma_{45}$$

$$X_7=(\Gamma_{12}\Gamma_{34}+\Gamma_{21}\Gamma_{34}-\Gamma_{24}\Gamma_{31}-\Gamma_{12}\Gamma_{31})\Gamma_{45}$$

$$Y_1=(\Gamma_{14}\Gamma_{21}+\Gamma_{24}\Gamma_{21}+\Gamma_{34}\Gamma_{21})\Gamma_{13}$$

$$Y_2=(\Gamma_{13}\Gamma_{14}+\Gamma_{14}\Gamma_{31}+\Gamma_{34}\Gamma_{13}+\Gamma_{34}\Gamma_{31})\Gamma_{45}$$

$$Z_1=(\Gamma_{14}+\Gamma_{24}+\Gamma_{34})\Gamma_{12}\Gamma_{31}$$

$$Z_2=(\Gamma_{14}\Gamma_{12}+\Gamma_{21}\Gamma_{14}+\Gamma_{12}\Gamma_{24}+\Gamma_{21}\Gamma_{24})\Gamma_{45}$$

$$Z_3=(\Gamma_{34}\Gamma_{12}+\Gamma_{21}\Gamma_{34}+\Gamma_{31}\Gamma_{14}+\Gamma_{31}\Gamma_{34})\Gamma_{45}$$

Text D: Explicit expressions of W_{jl}

$$W_{11}=-\kappa_{14}\frac{\Omega_c(\omega+d_{51})a_{431}^{*(2)}+\Omega_a^*(\omega+d_{31})\rho_{541}^{(2)}+(\omega+d_{31})(\omega+d_{51})(a_{111}^{(2)}-a_{441}^{(2)})}{D_1}, \quad \text{(D1)}$$

$$W_{12}=-\kappa_{14}\frac{\Omega_c(\omega+d_{51})a_{432}^{*(2)}+\Omega_a^*(\omega+d_{31})\rho_{542}^{(2)}+(\omega+d_{31})(\omega+d_{51})(a_{112}^{(2)}-a_{442}^{(2)}+a_{21}^{(2)})}{D_1}, \quad \text{(D2)}$$

$$W_{21}=-\kappa_{24}\frac{\Omega_c(\omega+d_{52})a_{431}^{*(2)}+\Omega_a^*(\omega+d_{32})\rho_{541}^{(2)}+(\omega+d_{32})(\omega+d_{52})(a_{221}^{(2)}-a_{441}^{(2)}+a_{21}^{*(2)})}{D_2}, \quad \text{(D3)}$$

$$W_{22}=-\kappa_{24}\frac{\Omega_c(\omega+d_{52})a_{432}^{*(2)}+\Omega_a^*(\omega+d_{32})\rho_{542}^{(2)}+(\omega+d_{32})(\omega+d_{52})(a_{222}^{(2)}-a_{442}^{(2)})}{D_2}, \quad \text{(D4)}$$

where $a_{21}^{(2)}=(a_{42}^{*(1)}-a_{41}^{(1)})/(\omega+d_{21})$.