Supplemental Material: Thermodynamics of the system of massive Dirac fermions in a uniform magnetic field

Ren-Hong Fang,[1] Ren-Da Dong,[2] De-Fu Hou,[2,*] and Bao-Dong Sun[1,†]

[1]*Key Laboratory of Particle Physics and Particle Irradiation (MOE), Institute of Frontier and Interdisciplinary Science, Shandong University, Qingdao 266237, China*

[2]*Institute of Particle Physics and Key Laboratory of Quark and Lepton Physics (MOS), Central China Normal University, Wuhan 430079, China*

I. EXPANSION OF $g(a,b,c)$ AT $b=0$

The function $g(a,b,c)$ for massive case is

$$\begin{aligned} g(a,b,c) \;=\; & \frac{b}{4\pi^2}\int_0^\infty ds[\ln(1+e^{a-\sqrt{s^2+c^2}})+\ln(1+e^{-a-\sqrt{s^2+c^2}})] \\ & +\frac{b}{2\pi^2}\int_0^\infty ds\sum_{n=1}^{\infty}[\ln(1+e^{a-\sqrt{nb+s^2+c^2}})+\ln(1+e^{-a-\sqrt{nb+s^2+c^2}})], \end{aligned} \tag{1}$$

where $a=\mu\beta$, $b=2eB\beta^2$, $c=m\beta$. Define an auxiliary function as

$$f(a,x)=\ln(1+e^{a-x})+\ln(1+e^{-a-x}), \tag{2}$$

then $g(a,b,c)$ in Eq. (1) becomes

$$g(a,b,c)=\frac{b}{2\pi^2}\int_0^\infty ds\left[\frac{1}{2}f(a,\sqrt{s^2+c^2})+\sum_{n=1}^{\infty}f(a,\sqrt{nb+s^2+c^2})\right]. \tag{3}$$

Defining another auxiliary function as

$$\mathcal{F}(z)=f(a,\sqrt{zb+s^2+c^2}), \tag{4}$$

and making use of following Abel-Plana formula,

*Electronic address: houdf@mail.ccnu.edu.cn

†Electronic address: sunbd@sdu.edu.cn

$$\frac{1}{2}\mathcal{F}(0)+\sum_{n=1}^{\infty}\mathcal{F}(n)=\int_0^{\infty}dt\mathcal{F}(t)+i\int_0^{\infty}dt\frac{\mathcal{F}(it)-\mathcal{F}(-it)}{e^{2\pi t}-1}, \tag{5}$$

we have

$$\begin{aligned} g(a,b,c) &= \frac{1}{2\pi^2}\int_0^{\infty}ds\int_0^{\infty}dtf(a,\sqrt{t+s^2+c^2}) \\ &+\frac{b}{2\pi^2}\times i\int_0^{\infty}ds\int_0^{\infty}dt\frac{1}{e^{2\pi t}-1}\left[f(a,\sqrt{itb+s^2+c^2})-f(a,\sqrt{-itb+s^2+c^2})\right]. \end{aligned} \tag{6}$$

We can see that the first term in $g(a,b,c)$ is irrelevant to b now. In the following we will focus on the second term.

Define a third auxiliary function as follows,

$$F(a,c,x)=\int_0^{\infty}dsf(a,\sqrt{x^2+s^2+c^2}), \tag{7}$$

then $g(a,b,c)$ in Eq. (6) becomes

$$g(a,b,c)=\frac{1}{2\pi^2}\int_0^{\infty}ds\int_0^{\infty}dtf(a,\sqrt{t+s^2+c^2})+\frac{b}{2\pi^2}\times i\int_0^{\infty}dt\frac{F(a,c,\sqrt{itb})-F(a,c,\sqrt{-itb})}{e^{2\pi t}-1}. \tag{8}$$

By the variable transformation $y=\sqrt{x^2+s^2}$ in Eq. (7), $F(a,c,x)$ can be rewritten as

$$F(a,c,x)=\int_{|x|}^{\infty}dy\frac{y}{\sqrt{y^2-x^2}}f(a,\sqrt{y^2+c^2}). \tag{9}$$

The factor $y/\sqrt{y^2-x^2}$ in the integrand in $F(a,x)$ can be replaced by following Taylor expansion,

$$\frac{y}{\sqrt{y^2-x^2}}=\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{x^{2n}}{y^{2n}}, \tag{10}$$

where we have defined $(-1)!!=0!!=1$. Then $F(a,x)$ becomes

$$F(a,c,x)\equiv\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^{2n}d_n(a,c,x), \tag{11}$$

where we have defined $d_n(a,c,x)$ as

$$d_n(a,c,x)=\int_{|x|}^{\infty}dy\frac{1}{y^{2n}}f(a,\sqrt{y^2+c^2}). \tag{12}$$

Since $d_n(a,c,x)=d_n(a,c,-x)$, the derivative of $d_n(a,c,x)$ with respect to x is

$$d_n'(a,c,x) = -\frac{|x|}{x^{2n+1}} f(a, \sqrt{x^2+c^2}). \tag{13}$$

We can expand $f(a, \sqrt{x^2+c^2})$ at $x=0$ as follows,

$$f(a, \sqrt{x^2+c^2}) = \sum_{k=0}^{\infty} w_{2k}(a,c) x^{2k}, \tag{14}$$

then $d_n(a,c,x)$ becomes

$$d_n(a,c,x) = |x| \sum_{k=0}^{\infty} w_{2k}(a,c) \frac{1}{2n-2k-1} x^{2k-2n} + C_n(a,c), \tag{15}$$

where $C_n(a,c)$ is independent of x. Now $F(a,c,x)$ in Eq. (11) becomes

$$F(a,c,x) = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} C_n(a,c) x^{2n}, \tag{16}$$

where we have used

$$\sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n-2k-1} = 0, \quad (k = 0, 1, 2, \cdots). \tag{17}$$

The coefficient $C_n(a,c)$ in Eq. (16) is just the constant term when $d_n(a,c,x)$ in expanded at $x=0$. When $n>0$, we can rewrite $d_n(a,c,x)$ in (12) through integration by parts as

$$\begin{aligned} d_n(a,c,x) &= \sum_{k=0}^{2n-2} \frac{(2n-k-2)!}{(2n-1)!} \frac{1}{x^{2n-k-1}} \frac{d^k}{dx^k} f(a, \sqrt{x^2+c^2}) - \frac{\ln x}{(2n-1)!} \frac{d^{2n-1}}{dx^{2n-1}} f(a, \sqrt{x^2+c^2}) \\ &\quad - \frac{1}{(2n-1)!} \int_x^{\infty} dy \ln y \frac{d^{2n}}{dy^{2n}} f(a, \sqrt{y^2+c^2}). \end{aligned} \tag{18}$$

which implies

$$C_n(a,c) = -\frac{1}{(2n-1)!} \int_0^{\infty} dy \ln y \frac{d^{2n}}{dy^{2n}} f(a, \sqrt{y^2+c^2}). \tag{19}$$

When $n=0$, we have

$$C_0(a,c) = \int_0^{\infty} dy f(a, \sqrt{y^2+c^2}). \tag{20}$$

Substituting Eq. (16) into Eq. (8) gives

$$g(a,b,c) = \frac{1}{2\pi^2} \int_0^{\infty} ds \int_0^{\infty} dt f(a, \sqrt{t+s^2+c^2}) - \frac{1}{\pi^2} \sum_{n=0}^{\infty} \frac{(4n+1)!!}{(4n+4)!!} \mathscr{B}_{2n+2} C_{2n+1}(a,c) b^{2n+2}, \tag{21}$$

where we have used following integrations,

$$\int_0^\infty dt \frac{t^{2n+1}}{e^{2\pi t}-1} = (-1)^n \frac{\mathscr{B}_{2n+2}}{4n+4}, \quad (n \geqslant 0), \tag{22}$$

with Bernoulli numbers $\mathscr{B}_n$ defined as

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{\mathscr{B}_n}{n!} t^n. \tag{23}$$

II. ASYMPTOTIC BEHAVIORS OF $C_n(a, c)$ $(n \geqslant 1)$ AT $c = 0$

The expression of $C_n(a, c)$ $(n \geqslant 1)$ in Eq. (19) is

$$C_n(a, c) = -\frac{1}{(2n-1)!} \int_0^\infty dy \ln y \frac{d^{2n}}{dy^{2n}} f(a, \sqrt{y^2 + c^2}), \tag{24}$$

where $f(a, x) = \ln(1 + e^{a-x}) + \ln(1 + e^{-a-x})$. Making use of integration by parts, Eq. (24) becomes

$$\begin{aligned} C_n(a, c) &= \frac{1}{(2n-1)!} \int_0^\infty dy \ln y \frac{d^{2n-1}}{dy^{2n-1}} \left[\left(\frac{1}{e^{\sqrt{y^2+c^2}-a} + 1} + \frac{1}{e^{\sqrt{y^2+c^2}+a} + 1} \right) \frac{y}{\sqrt{y^2 + c^2}} \right] \\ &= -\frac{1}{(2n-1)!} \int_0^\infty dy \frac{1}{y} \frac{d^{2n-2}}{dy^{2n-2}} \left[\left(\frac{1}{e^{\sqrt{y^2+c^2}-a} + 1} + \frac{1}{e^{\sqrt{y^2+c^2}+a} + 1} \right) \frac{y}{\sqrt{y^2 + c^2}} \right], \end{aligned} \tag{25}$$

where the second equal sign is valid for $n > 1$ with an arbitrary c, and for $n = 1$ with $c \neq 0$.

When $c = 0$, Eq. (25) becomes

$$C_n(a, 0) = -\frac{1}{(2n-1)!} \int_0^\infty dy \frac{1}{y} \frac{d^{2n-2}}{dy^{2n-2}} \left(\frac{1}{e^{y-a} + 1} + \frac{1}{e^{y+a} + 1} \right). \tag{26}$$

The expression in the bracket in the integrand can be expanded at $y = 0$ as follows,

$$\frac{1}{e^{y-a} + 1} + \frac{1}{e^{y+a} + 1} = 1 + \#y + \#y^3 + \#y^5 + \cdots, \tag{27}$$

with "#" representing some coefficients. The expansion in Eq. (27) implies that, $C_n(a, 0)$ is divergent for $n = 1$, and convergent for $n > 1$. Further analysis shows that

$$\left. \frac{\partial}{\partial c} C_n(a, c) \right|_{c \to 0} = \infty. \tag{28}$$

We may conclude that, the asymptotic behaviors of $C_n(a, c)$ at $c = 0$ is

$$C_1(a, c) \sim \# \ln c + (\text{terms regular at } c = 0), \tag{29}$$

$$C_n(a, c) \sim \# c \ln c + (\text{terms regular at } c = 0), \quad (n > 1). \tag{30}$$

III. EXPANSION OF $C_1(a,c)$ AT $c=0$

When $n=1$, Eq. (25) becomes

$$C_1(a,c) = -\int_0^\infty dy \frac{1}{\sqrt{y^2+c^2}} \left(\frac{1}{e^{\sqrt{y^2+c^2}-a}+1} + \frac{1}{e^{\sqrt{y^2+c^2}+a}+1} \right). \tag{31}$$

Making use of the variable transformation, $x=(y^2+c^2)^{1/2}$, the expression of $C_1(a,c)$ in Eq. (31) is equivalent to

$$C_1(a,c) = -\int_{|c|}^\infty dx \frac{1}{x} \left(1-\frac{c^2}{x^2}\right)^{-\frac{1}{2}} \left(\frac{1}{e^{x-a}+1} + \frac{1}{e^{x+a}+1} \right). \tag{32}$$

Since $x^2 > c^2$, we can use the Taylor expansion for $(1-c^2/x^2)^{-1/2}$ in Eq. (10), leading to

$$C_1(a,c) = -\sum_{n=0}^\infty \frac{(2n-1)!!}{(2n)!!} c^{2n} X_n(a,c), \tag{33}$$

where $X_n(a,c)$ can be written as follows,

$$X_n(a,c) = \int_{|c|}^\infty dx \left[\frac{1}{x^{2n+1}} \left(\frac{1}{e^{x-a}+1} + \frac{1}{e^{x+a}+1} - 1 \right) + \frac{1}{x^{2n+1}} \right]. \tag{34}$$

The derivative of $X_n(a,c)$ with respect to c is

$$X_n'(a,c) = -\frac{|c|}{c^{2n+2}} \left(\frac{1}{e^{c-a}+1} + \frac{1}{e^{c+a}+1} - 1 \right) - \frac{1}{c^{2n+1}}. \tag{35}$$

The term in the bracket in Eq. (35) is an odd function of c which can be expanded at $c=0$ as

$$\frac{1}{e^{c-a}+1} + \frac{1}{e^{c+a}+1} - 1 = \sum_{k=0}^\infty J_{2k+1}(a) c^{2k+1}, \tag{36}$$

then $X_n(a,c)$ can be obtained from $X_n'(a,c)$,

$$X_n(a,c) = -|c| \sum_{k=0}^\infty J_{2k+1}(a) \frac{1}{2k-2n+1} c^{2k-2n} + \begin{cases} D_0(a) - \frac{1}{2}\ln c^2, & n=0 \\ D_n(a) + \frac{1}{2n} c^{-2n}, & n>0 \end{cases}, \tag{37}$$

where $D_n(a)$ are independent of c and can be determined by the same method as the calculation of $C_n(a,c)$ in Sec. I. The result of $D_n(a)$ is

$$D_n(a) = -\frac{1}{(2n)!} \int_0^\infty dx \ln x \frac{d^{2n+1}}{dx^{2n+1}} \left(\frac{1}{e^{x-a}+1} + \frac{1}{e^{x+a}+1} \right). \tag{38}$$

Taking use of

$$\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n-2k-1}=0,\quad (k=0,1,2,\cdots), \tag{39}$$

$$\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{n}=\ln 4, \tag{40}$$

we get

$$C_1(a,c)=\frac{1}{2}\ln c^2-[D_0(a)+\ln 2]-\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}D_n(a)c^{2n}. \tag{41}$$

which is consistent with the asymptotic analysis in Eq. (29).

According to the Appendix D in [1], $D_n(a)$ can be expanded at $a=0$ as follows,

$$D_n(a)=(-\ln 4-\gamma)\delta_{n,0}-\frac{2}{(2n)!}\sum_{k=0}^{\infty}\left(2^{2n+2k+1}-1\right)\zeta'(-2n-2k)\frac{a^{2k}}{(2k)!}. \tag{42}$$

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[1] C. Zhang, R.-H. Fang, J.-H. Gao, and D.-F. Hou, Phys. Rev. D **102**, 056004 (2020), 2005.08512.