Supplemental Material for "PT Symmetry Induced Rings of Lasing Threshold Modes Embedded with Discrete Bound States in the Continuum"

Qianju Song(宋前举)[1,†], Shiwei Dai(戴士为)[1,†], Dezhuan Han(韩德专)[1,*], Z. Q. Zhang(张昭庆)[2], C. T. Chan(陈子亭)[2,*], and Jian Zi(资剑)[3,*]

[1] College of Physics, Chongqing University, Chongqing 401331, China

[2] Department of Physics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong, China

[3] Department of Physics, Key Laboratory of Micro- and Nano-Photonic Structures (MOE), and State Key Laboratory of Surface Physics, Fudan University, Shanghai 200433, China

[*]Corresponding authors.

E-mail addresses: dzhan@cqu.edu.cn (D. Z. Han), phchan@ust.hk (C. T. Chan), and jzi@fudan.edu.cn (J. Zi)

[†] These authors contributed equally to this work.

(I) The equality between γ_{tot} and $-\omega''$

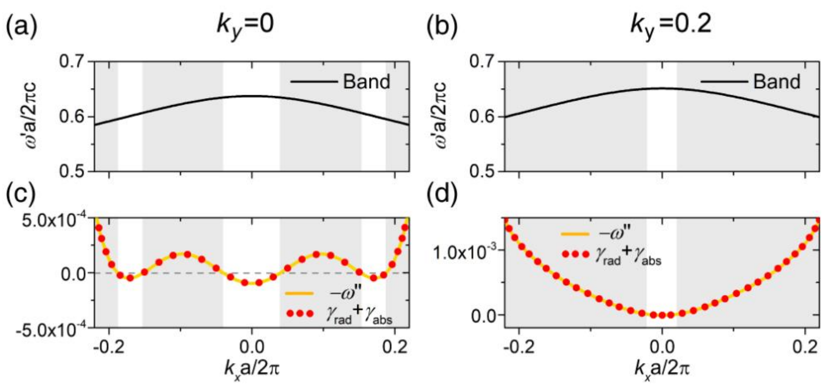

Fig. S1. Examples of simulated dispersion, imaginary part ω'' and total decay rate $\gamma_{\text{tot}}=\gamma_{\text{rad}}+\gamma_{\text{abs}}$. The wave vectors are chosen k_y=0 and k_y=0.2 in the left and right panels, respectively. The total decay rate $\gamma_{\text{tot}}=\gamma_{\text{rad}}+\gamma_{\text{abs}}$ coincides with $-\omega''$.

In Fig. 1(e) of the main text, we only show the imaginary part of the eigenfrequency ω'' and claim that total decay rate $\gamma_{\text{tot}}=\gamma_{\text{rad}}+\gamma_{\text{abs}}$ and $-\omega''$ of the resonance modes are coincident. Here, a numerical verification of γ_{tot} always coincide with $-\omega''$ is shown in Fig. S1. We take the resonance modes on the TE-like band but with different wave vectors as an example. Dispersion of TE-like band with k_y=0 and k_y=0.2 are shown in Figs. S1(a) and S1(b), respectively. In Figs. S1(c) and S1(d), we plot $-\omega''$ and the total decay rate γ_{tot} of the resonance modes, which is coincides with each other.

(II) *Pt*-BICs and lasing threshold modes at small PT-symmetric perturbation

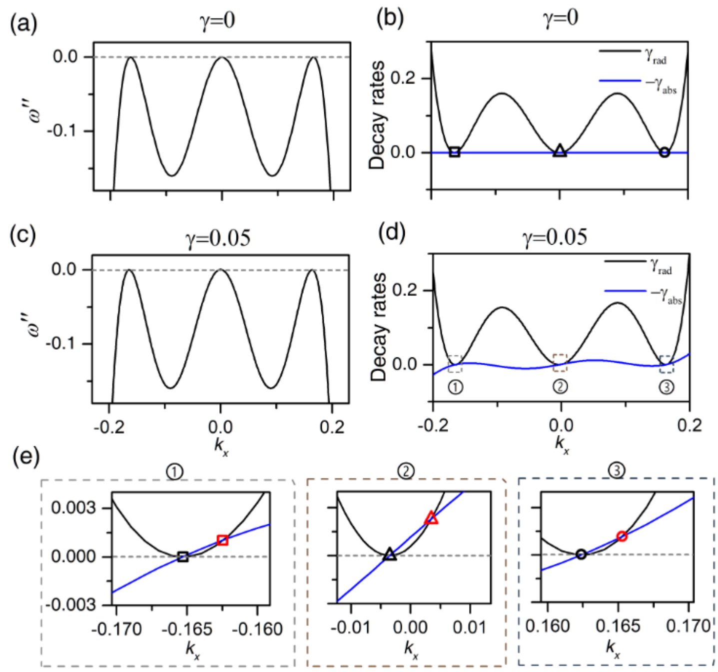

Fig. S2. Decay rates calculated from COMSOL Multiphysics. The imaginary part, absorption and radiation rate of resonance modes for γ=0 (a-b) and γ=0.05 (c-e). The intersections of curve γ_{rad} and $-\gamma_{\text{abs}}$ with $\gamma_{\text{rad}}=\gamma_{\text{abs}}=0$ correspond to *pt*-BICs (black symbols). The other three with finite γ_{rad} correspond to the lasing threshold modes (red symbols). Three zoom-in boxes in (e) correspond to the open rectangular regions shown in (d).

When a small PT-symmetric perturbation is introduced into the system, the splitting of an ordinary BIC into a pair of *pt*-BIC and a lasing threshold mode also exists. For a system without gain and loss, there exists three *pt*-BICs with $\omega''=0$ as shown in Fig. S2(a). The radiation rate γ_{rad} is semi-positive definite and the absorption rate γ_{abs} along the k_x axis is always zero, therefore, γ_{rad}=0 correspond to BICs. When the system is perturbed by PT symmetry, γ_{abs} fluctuates around zero along the k_x axis. γ_{abs} can intersect with the curves of γ_{rad} at other three points as shown in Fig. S2(c-e), which is close to three BICs. The three intersections with $\gamma_{\text{rad}}=-\gamma_{\text{abs}}\neq 0$ correspond to the lasing threshold modes (red symbols in Fig. S2(e)), at which the radiation loss is exactly balanced by the net gain.

(III) Behaviors of Q factor near other *pt*-BICs

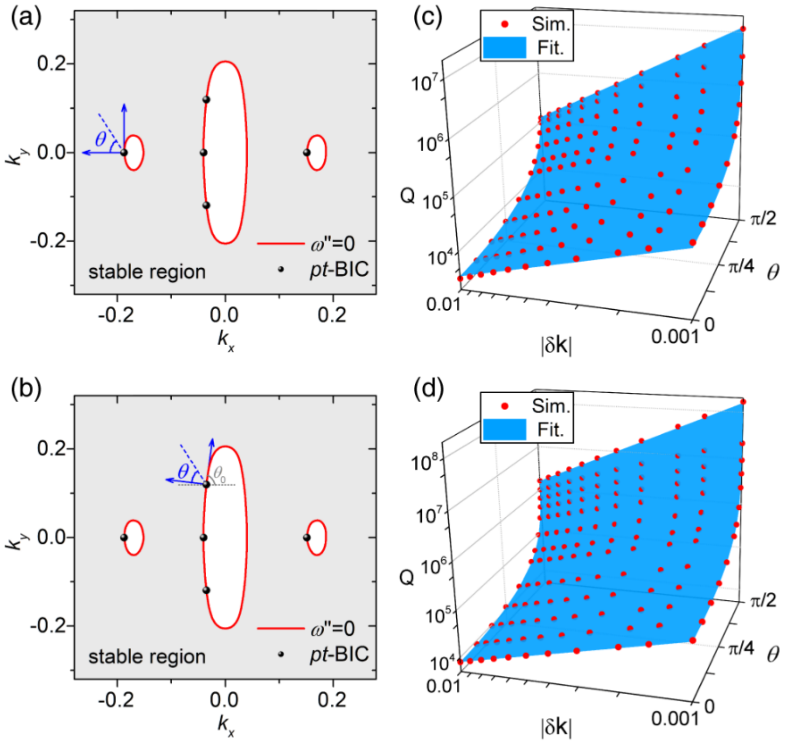

Fig. S3. (a-b) Lasing threshold modes (red lines) and *pt*-BICs (black dots) on the TE-like band as shown in the k_x-k_y plane. The Q factors near *pt*-BICs point at (k_{x0}, k_{y0})=(–0.188, 0) and (–0.035, 0.112) are shown in (c) and (d), respectively. (c) For $\theta=\pi/2$, the Q factor diverges at the rate of δk_y^{-2}, while at the rate of δk_x^{-1} for $\theta=0$. (d) When the *pt*-BIC appear at off-high symmetry lines, the Q factor diverges at the rate of $[\delta k_x \cos\theta_0 + \delta k_y \sin\theta_0]^{-2}$ along the tangential direction of the corresponding ring at the point of *pt*-BIC. The divergence rate of Q factor becomes $[-\delta k_x \sin\theta_0 + \delta k_y \cos\theta_0]^{-1}$ along the normal direction (θ=0).

Here, we also show the divergence behaviors of Q factor near the other *pt*-BICs. In Fig. S3(a), we take the *pt*-BIC point at (k_{x0}, k_{y0})=(–0.188, 0) as the origin of coordinates and θ is the included angle between negative *x* direction and dashed line. For any arbitrary direction, we define $\delta k_x=k_x-k_{x0}=|\delta k|\cos\theta$ and $\delta k_y= k_y-k_{y0} =|\delta k|\sin\theta$. The Q factors along the dashed line with different included angle θ close to *pt*-BIC are shown in Fig. S3(c). For θ=0, ω'' crosses zero linearly near a *pt*-BIC and $\partial\omega''/\partial k_x$ is not zero. Therefore, $\omega'' \propto \delta k_x$ and the Q factor is proportional to δk_x^{-1}. However, for $\theta=\pi/2$, the linear term vanishes and $\omega'' \propto \delta k_y^2$, hence the Q factor carries a completely different Q-factor divergence rate, i.e., $\mathrm{Q} \propto \delta k_y^{-2}$. The divergence rate of Q factor has the form of $[c\delta k_x + \delta k_y^2]^{-1}$, where *c* is a constant to be determined.

By fitting the simulated Q factor of resonance modes with this formula, we can extract the coefficient c=0.35.

The *pt*-BIC at off-high symmetry lines also holds similar anisotropic behavior. For convenience, we take the *pt*-BIC point at (k_{x0}, k_{y0})=(–0.035, 0.112) as origin of coordinates as shown in Figs. S3(b). The tangential and normal directions of a ring of lasing threshold modes are defined as the coordinate axes of a local coordinate system. The local coordinate and the k_x-k_y coordinate system can be linked by rotation operation and we obtain θ_0=0.472π. The Q factors along the dashed line with different θ close to the *pt*-BICs are shown in Fig. S3(d). For $\theta=\pi/2$, the Q factor is proportional to $[\delta k_x\cos\theta_0+\delta k_y\sin\theta_0]^{-2}$. For the other direction, the divergence rate of Q factor becomes $[-\delta k_x\sin\theta_0+\delta k_y\cos\theta_0]^{-1}$. Thus, the divergence rate of Q factor has the generic form of $[c(-\delta k_x\sin\theta_0+\delta k_y\cos\theta_0)+(\delta k_x\cos\theta_0+\delta k_y\sin\theta_0)^2]^{-1}$, where c is a constant to be determined. By fitting the simulated Q factor of resonance modes with this formula, we can extract the coefficient c=1.95.