Supplemental Material: Two-dimensional quantum walk with non-Hermitian skin effects

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I. THE EFFECTIVE HAMILTONIAN IN MOMENTUM SPACE

The explicit form of the time-dependent, effective Hamiltonian $H(\mathbf{k},t)$ in Eq. (2) is constructed as the following

$$H(\mathbf{k},t)=\left(\begin{bmatrix}0 & b(t)e^{-ik_x}+d(t)e^{-ik_y}\\ a(t)e^{-ik_x}+c(t)e^{-ik_y} & 0\end{bmatrix}+H.c.\right)+\begin{bmatrix}\gamma(t) & 0\\ 0 & -\gamma(t)\end{bmatrix}, \tag{S1}$$

with the step-wise, time-dependent parameters [1]

$$a(t)=i(\frac{\pi}{2}-\theta_1)G(t-\frac{1}{10}), \tag{S2}$$

$$b(t)=-i\theta_2G(t-\frac{3}{10}), \tag{S3}$$

$$\gamma(t)=i\gamma G(t-\frac{5}{10}), \tag{S4}$$

$$c(t)=i\theta_2G(t-\frac{7}{10}), \tag{S5}$$

$$d(t)=i\theta_1G(t-\frac{9}{10}), \tag{S6}$$

$$G(t)=5\Theta(t+\frac{1}{10})\Theta(\frac{1}{10}-t). \tag{S7}$$

Here $\Theta(t)$ is the Heaviside step function. The time-dependent coefficients divide one Floquet period into five segments, each implementing a gate operation.

II. GENERALIZED BRILLOUIN ZONE

We now outline the recipe for calculating the generalized Brillouin zone, which is encoded in $\beta(p_x,k_y,t)$ under open boundary conditions. Following Eq. (3) in the main text, we have

$$U_\epsilon(\mathbf{k},t)=\begin{cases}e^{-iH_1(\mathbf{k})10t}, & 0\le t\le\frac{1}{10}\\ e^{-iH_2(\mathbf{k})10(t-\frac{1}{10})}e^{-iH_1(\mathbf{k})}, & \frac{1}{10}<t\le\frac{2}{10}\\ e^{-im(\mathbf{k})10(t-\frac{2}{10})}e^{-iH_2(\mathbf{k})}e^{-iH_1(\mathbf{k})}, & \frac{2}{10}<t\le\frac{3}{10}\\ e^{-iH_3(\mathbf{k})10(t-\frac{3}{10})}e^{-im(\mathbf{k})}e^{-iH_2(\mathbf{k})}e^{-iH_1(\mathbf{k})}, & \frac{3}{10}<t\le\frac{4}{10}\\ e^{-iH_4(\mathbf{k})10(t-\frac{4}{10})}e^{-iH_3(\mathbf{k})}e^{-im(\mathbf{k})}e^{-iH_2(\mathbf{k})}e^{-iH_1(\mathbf{k})}, & \frac{4}{10}<t\le\frac{5}{10}\\ e^{-iH_\epsilon^{\text{eff}}(\mathbf{k})(2-2t)}, & \frac{1}{2}<t\le 1\end{cases} \tag{S8}$$

where

$$H_1(\mathbf{k}) = (\frac{\pi}{2} - \theta_1)\begin{pmatrix} 0 & -ie^{ik_x} \\ ie^{-ik_x} & 0 \end{pmatrix}, \tag{S9}$$

$$H_2(\mathbf{k}) = -\theta_2\begin{pmatrix} 0 & ie^{-ik_x} \\ -ie^{ik_x} & 0 \end{pmatrix}, \tag{S10}$$

$$m(\mathbf{k}) = i\gamma\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{S11}$$

$$H_3(\mathbf{k}) = \theta_2\begin{pmatrix} 0 & -ie^{ik_y} \\ ie^{-ik_y} & 0 \end{pmatrix}, \tag{S12}$$

$$H_4(\mathbf{k}) = \theta_1\begin{pmatrix} 0 & ie^{-ik_y} \\ -ie^{ik_y} & 0 \end{pmatrix}. \tag{S13}$$

The step-wise $H(\mathbf{k}, t)$ and $U_\epsilon(\mathbf{k}, t)$ lead to a time-dependent $\beta(p_x, k_y, t)$ that is also step-wise in its functional form. As an example, we show the calculation β for $t \in (\frac{2}{5}, \frac{1}{2}]$.

With open boundaries in the x direction, k_y is still a good quantum number, and system is reduced to a one-dimensional quantum walk with an additional parameter k_y. Fourier transforming Eq. (S8) over k_x, we find the time-period operator for the reduced one-dimensional quantum walk within the time interval $t \in (\frac{2}{5}, \frac{1}{2}]$

$$U_\epsilon(k_y, t) = \sum_m |m, A\rangle\langle m, A| \otimes \bar{A} + |m, B\rangle\langle m, B| \otimes \bar{B} + \tag{S14}$$

$$|m, A\rangle\langle m, B| \otimes \bar{C} + |m, B\rangle\langle m, A| \otimes \bar{D} + \tag{S15}$$

$$|m, B\rangle\langle m+1, A| \otimes \bar{E} + |m+1, A\rangle\langle m, B| \otimes \bar{F} + \tag{S16}$$

$$|m, A\rangle\langle m+1, A| \otimes \bar{G} + |m+1, A\rangle\langle m, A| \otimes \bar{H} + \tag{S17}$$

$$|m-1, B\rangle\langle m, B| \otimes \bar{J} + |m, B\rangle\langle m-1, B| \otimes \bar{K}, \tag{S18}$$

where $|m, A\rangle$ labels sublattice site A of the m the unit cell along x, and

$$\bar{A} = \sin\theta_1 \cos\theta_2 A_1 A_2 M_1, \tag{S19}$$

$$\bar{B} = \sin\theta_1 \cos\theta_2 B_1 B_2 M_2, \tag{S20}$$

$$\bar{C} = A_1 A_2 M_1(-\cos\theta_1 \cos\theta_2 P_1 + \sin\theta_1 \sin\theta_2 P_0), \tag{S21}$$

$$\bar{D} = B_1 B_2 M_2(\cos\theta_1 \cos\theta_2 P_1 - \sin\theta_1 \sin\theta_2 P_0), \tag{S22}$$

$$\bar{E} = B_1 B_2 M_2(-\cos\theta_1 \cos\theta_2 P_0 + \sin\theta_1 \sin\theta_2 P_1), \tag{S23}$$

$$\bar{F} = A_1 A_2 M_1(\cos\theta_1 \cos\theta_2 P_0 - \sin\theta_1 \sin\theta_2 P_1), \tag{S24}$$

$$\bar{G} = A_1 A_2 M_1(-\cos\theta_1 \sin\theta_2 P_0), \tag{S25}$$

$$\bar{H} = A_1 A_2 M_1(-\cos\theta_1 \sin\theta_2 P_1), \tag{S26}$$

$$\bar{J} = B_1 B_2 M_2(-\cos\theta_1 \sin\theta_2 P_1), \tag{S27}$$

$$\bar{K} = B_1 B_2 M_2(-\cos\theta_1 \sin\theta_2 P_0) \tag{S28}$$

with

$$
\begin{aligned}
M_1 &= \begin{pmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}, M_2 = \begin{pmatrix} e^{-\gamma} & 0 \\ 0 & e^{\gamma} \end{pmatrix}, P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
A_1 &= \begin{pmatrix} \cos\theta_1^{'} & \sin\theta_1^{'} e^{-ik_y} \\ -\sin\theta_1^{'} e^{ik_y} & \cos\theta_1^{'} \end{pmatrix}, B_1 = \begin{pmatrix} \cos\theta_1^{'} & -\sin\theta_1^{'} e^{ik_y} \\ \sin\theta_1^{'} e^{-ik_y} & \cos\theta_1^{'} \end{pmatrix}, \\
A_2 &= \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 e^{ik_y} \\ \sin\theta_2 e^{-ik_y} & \cos\theta_2 \end{pmatrix}, B_2 = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 e^{-ik_y} \\ -\sin\theta_2 e^{ik_y} & \cos\theta_2 \end{pmatrix}.
\end{aligned} \tag{S29}
$$

Here $\theta_1^{'} = (10t-4)\theta_1$, for $t \in (\frac{2}{5}, \frac{1}{2}]$.

Assuming a bulk state ansatz

$$
|\psi\rangle = \sum_m \beta^{-2m}(|m, A\rangle \otimes |\phi\rangle + \beta^{-1}|m, B\rangle \otimes \sigma_x|\phi\rangle), \tag{S30}
$$

we have

$$
U_\epsilon(k_y, t)|\psi\rangle = \sum_m \beta^{-2m}(|m, A\rangle \otimes (\bar{A} + \beta^{-1}\bar{C}\sigma_x) + |m, B\rangle \otimes (\bar{D} + \beta^{-1}\bar{B}\sigma_x) + |m+1, A\rangle \otimes (\bar{H} + \beta^{-1}\bar{F}\sigma_x) \tag{S31}
$$

$$
+ |m+1, B\rangle \otimes (\beta^{-1}\bar{K}\sigma_x) + |m-1, B\rangle \otimes (\bar{E} + \beta^{-1}\bar{J}\sigma_x) + |m-1, A\rangle \otimes \bar{G})|\phi\rangle \tag{S32}
$$

$$
= \lambda \sum_m \beta^{-2m}(|m, A\rangle \otimes |\phi\rangle + \beta^{-1}|m, B\rangle \otimes \sigma_x|\phi\rangle). \tag{S33}
$$

Non-trivial solution of the eigen equation above exists only if

$$
\det(\bar{A} - \lambda + \beta^{-1}\bar{C}\sigma_x + \beta^{-2}\bar{G} + \beta\bar{F}\sigma_x + \beta^2\bar{H}) = 0. \tag{S34}
$$

At any given instant within the time range $(\frac{2}{5}, \frac{1}{2}]$, we numerically solve the eigen spectum λ of an open chain, from which we get four non-zero solutions β using Eq. (S34). We sort these solutions as $|\beta_1| \le |\beta_2| \le |\beta_3| \le |\beta_4|$. The condition of $|\beta_2| = |\beta_3|$ fixes the generalized Brillouin zone [2].

The calculation of β in other time intervals are similar. In particular, for $t \in (\frac{1}{2}, 1]$, β is independent of time.

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