Supplemental Material for: "Ideal spin hydrodynamics from Wigner function approach"

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Here we provide details of derivations for some important equations in the main text.

I. THE WIGNER FUNCTION

In absence of particle scatterings, the Wigner function satisfies the following kinetic equation

$$\left[\gamma_\mu\left(p^\mu+\frac{i}{2}\partial^\mu\right)-m\right]W(x,p)=0\,. \tag{1}$$

Multiplying this equation with $\gamma\cdot p+(i/2)\gamma\cdot\partial+m$ from the left, we obtain

$$\left[\left(p^2-\frac{1}{4}\partial^2-m^2\right)+ip\cdot\partial\right]W(x,p)=0\,. \tag{2}$$

The mass-shell condition and Vlasov equation for the Wigner function are obtained by linear recombinations of Eq. (2) and its Hermitian conjugate,

$$\begin{aligned}\left(p^2-\frac{1}{4}\partial^2-m^2\right)W(x,p)&=0\,,\\ p\cdot\partial W(x,p)&=0\,,\end{aligned} \tag{3}$$

where we have used the property $W^\dagger=\gamma^0 W\gamma^0$. From the mass-shell condition we find that the momentum p^μ for the Wigner function is no longer on the traditional mass-shell $p^2=m^2$ at second order in space-time gradient.

To solve the Wigner function, we make a gradient expansion as

$$W(x,p)=\sum_{i=0,1,2}W_i(x,p)+\mathcal{O}(\mathrm{Kn}^3)\,, \tag{4}$$

where the subscript $i=0,1,2$ labels orders in space-time gradient. We note that the gradient expansion for the Wigner function is equivalent to an expansion with respect to the Knudsen number Kn, thus here and in the follows we use Kn to label orders in the gradient expansion. The zeroth order part satisfies the following equations

$$\begin{aligned}(\gamma\cdot p-m)W_0(x,p)&=0\,,\\ (p^2-m^2)W_0(x,p)&=0\,,\\ p\cdot\partial W_0(x,p)&=0\,,\end{aligned} \tag{5}$$

which are obtained from leading-order terms of Eqs. (1) and (3). A general solution of W_0 can be constructed by directly inserting the quantized field operators $\psi(x)$ and $\bar\psi(x)$ into the definition of the Wigner function, c.f., $W_0(x,p)$ in Eq. (4) in the main text. We then assume that higher order terms of the Wigner function are related to W_0 and take the trial solution as

$$W(x,p)=UW_0(x,p)\gamma^0U^\dagger\gamma^0+\mathcal{O}(\mathrm{Kn}^3)\,, \tag{6}$$

where the matrix operator U is assumed to be function of derivative operator

$$U\equiv 1+U_1\gamma\cdot\partial+U_2\partial^2+\mathcal{O}(\mathrm{Kn}^3) \tag{7}$$

In order to fulfill Eqs. (1), W_1 should satisfy

$$(\gamma\cdot p-m)W_1(x,p)+\frac{i}{2}\gamma\cdot\partial W_0(x,p)=0\,. \tag{8}$$

According to the assumption (6), W_1 is given by

$$W_1(x,p)=U_1\gamma\cdot\partial W_0(x,p)+W_0(x,p)\gamma\cdot\overleftarrow{\partial}\gamma^0U_1^\dagger\gamma^0\,. \tag{9}$$

Inserting W_1 into Eq. (8), we obtain

$$\begin{aligned}0=&(\gamma\cdot p-m)U_1\gamma\cdot\partial W_0(x,p)+\frac{i}{2}\gamma\cdot\partial W_0(x,p)\\ &+[(\gamma\cdot p-m)W_0(x,p)]\,\gamma\cdot\overleftarrow{\partial}\gamma^0U_1^\dagger\gamma^0\,.\end{aligned} \tag{10}$$

We then use the following relation to commute $\gamma\cdot p-m$ with $U_1\gamma\cdot\partial$ so that the result can be simplified with the first line in Eq. (5),

$$\begin{aligned}&(\gamma\cdot p-m)U\gamma\cdot\partial\\ &=\{\gamma\cdot p,U\gamma\cdot\partial\}-U\gamma\cdot\partial(\gamma\cdot p-m)-2mU\gamma\cdot\partial\,.\end{aligned} \tag{11}$$

Finally we obtain an equation for U_1,

$$\begin{aligned}U_1\gamma\cdot\partial W_0(x,p)=&\frac{i}{4m}\gamma\cdot\partial W_0(x,p)\\ &+\frac{1}{2m}\left\{\gamma\cdot p,\,U_1\gamma\cdot\partial\right\}W_0(x,p)\,,\end{aligned} \tag{12}$$

which can be solved by iterative method and the result reads

$$U_1=\frac{i}{4m}\,. \tag{13}$$

Inserting U_1 into Eq. (6) and taking out the second-order parts, we obtain

$$W_2 = \frac{1}{16m^2}\gamma\cdot\partial W_0\gamma\cdot\overleftarrow{\partial} + \partial^2\left[U_2W_0 + \partial^2 W_0\gamma^0U_2^\dagger\gamma^0\right], \tag{14}$$

which should fulfills Eq. (1),

$$(\gamma\cdot p - m)W_2(x,p) + \frac{i}{2}\gamma\cdot\partial W_1(x,p) = 0\,. \tag{15}$$

The calculation of the left-hand-side of Eq. (15) is straightforward

$$\begin{aligned}&(\gamma\cdot p - m)W_2(x,p) + \frac{i}{2}\gamma\cdot\partial W_1(x,p)\\ &= \frac{1}{16m^2}(\gamma\cdot p - m)\gamma\cdot\partial W_0\gamma\cdot\overleftarrow{\partial}\\ &+(\gamma\cdot p - m)\partial^2\left[U_2W_0 + W_0\gamma^0U_2^\dagger\gamma^0\right]\\ &-\frac{1}{8m}\partial^2W_0 + \frac{1}{8m}\gamma\cdot\partial W_0\gamma\cdot\overleftarrow{\partial}\,.\end{aligned} \tag{16}$$

We further use the following relation

$$\begin{aligned}&(\gamma\cdot p - m)\gamma\cdot\partial\\ &= \{\gamma\cdot p, \gamma\cdot\partial\} - \gamma\cdot\partial(\gamma\cdot p - m) - 2m\gamma\cdot\partial\\ &= 2p\cdot\partial - \gamma\cdot\partial(\gamma\cdot p - m) - 2m\gamma\cdot\partial\,,\end{aligned} \tag{17}$$

together with the first line in Eq. (5) and obtain

$$(\gamma\cdot p - m)\partial^2\left[U_2W_0 + W_0\gamma^0U_2^\dagger\gamma^0\right] = \frac{1}{8m}\partial^2W_0\,. \tag{18}$$

The matrix U_2 must have a singularity at $p^2 = m^2$, otherwise the left-hand-side of Eq. (18) vanish because $(\gamma\cdot p - m)W_0 = 0$ for an on-shell momentum p^μ. A simplest solution of Eq. (18) reads

$$U_2W_0 + W_0\gamma^0U_2^\dagger\gamma^0 = \frac{\gamma\cdot p + m}{8m(p^2 - m^2)}W_0\,, \tag{19}$$

with

$$U_2 = \frac{\gamma\cdot p + m}{16m(p^2 - m^2)}\,. \tag{20}$$

Note that $\gamma^0U_2^\dagger\gamma^0 = U_2$ and find $\{U_2, W_0\} = 0$, where the later one can be proved using the explicit expression of W_0. Inserting U_1 in Eq. (13) and U_2 in Eq. (20) into Eqs. (6) and (7), one obtains first order and second order parts of the Wigner function, c.f., Eq. (6) in the main text.

II. HYDRODYNAMICAL QUANTITIES IN LOCAL EQUILIBRIUM

This section we will show details for deriving the current density J^μ, the energy-momentum tensor (density) $T^{\mu\nu}$, the spin tensor (density) $S^{\lambda,\mu\nu}$, and the dipole-moment tensor (density) $D^{\mu\nu}$.

A. Wigner function

In order to calculate the physical quantities, we first look at the Wigner function $W = W_0 + \delta W$, where W_0 is the zeroth order part and δW contains the first and second order parts of the Wigner function. The explicit expressions of W_0 and δW are given in Eqs. (4) and (6) in the main text. One can easily derive the zeroth order part W_0 using explicit expressions of wave functions $u_r(\mathbf{p})$ and $v_r(-\mathbf{p})$,

$$W_0 = \frac{1}{4}(m + \gamma\cdot p)(V + \gamma^5\gamma^\mu n_\mu)\delta(p^2 - m^2)\,, \tag{21}$$

where we have defined

$$\begin{aligned}V(x,p) &= \frac{2}{(2\pi)^3}\sum_{rs}\delta_{rs}\\ &\quad\times\left\{\theta(p^0)f^+_{rs}(x,\mathbf{p}) + \theta(-p^0)f^-_{rs}(x,-\mathbf{p})\right\}\,,\\ n^\mu(x,p) &= \frac{1}{(2\pi)^3m}\sum_{rs}\left\{\theta(p^0)\bar{u}_s(\mathbf{p})\gamma^\mu\gamma^5u_r(\mathbf{p})f^+_{rs}(x,\mathbf{p})\right.\\ &\quad\left.-\theta(-p^0)\bar{v}_s(-\mathbf{p})\gamma^\mu\gamma^5v_r(-\mathbf{p})f^-_{rs}(x,-\mathbf{p})\right\}.\end{aligned} \tag{22}$$

With the help of W_0, the calculation of δW is then straightforward with Eq. (6) in the main text. Using Eq. (9) in the main text, we obtain the following results in local thermal equilibrium

$$\begin{aligned}V_{\rm eq} &= \frac{4}{(2\pi)^3}\left(1 + \frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\right)\\ &\quad\times\left[\theta(p^0)\exp(-\beta\cdot p + \xi) + \theta(-p^0)\exp(\beta\cdot p - \xi)\right]\,,\\ n^\mu_{\rm eq} &= -\frac{1}{(2\pi)^3m}\epsilon^{\mu\nu\alpha\beta}p_\nu\omega_{\alpha\beta}\\ &\quad\times\left[\theta(p^0)\exp(-\beta\cdot p + \xi) - \theta(-p^0)\exp(\beta\cdot p - \xi)\right]\,,\end{aligned} \tag{23}$$

where we have taken the Boltzmann limit and truncated the expansion series at $\mathcal{O}(\chi_s^2)$.

B. Current density and energy-momentum tensor

The current density J^μ and the energy-momentum tensor (density) $T^{\mu\nu}$ are given by the vector component of the Wigner function as shown as Eq. (11) and (12) in the main text. Here $\mathcal{V}^\mu$ is extracted from W by

$$\mathcal{V}^\mu = \mathrm{Tr}\left[\gamma^\mu(W_0 + \delta W)\right]\,. \tag{24}$$

The vector components of W_0 and δW are respectively given by

$$\begin{aligned}&\mathrm{Tr}\,(\gamma^\mu W_0) = \delta(p^2 - m^2)p^\mu V(x,p)\,,\\ &\mathrm{Tr}\,(\gamma^\mu\delta W) = -\delta(p^2 - m^2)\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\alpha\partial_\nu n_\beta(x,p)\\ &\qquad-\left[\frac{\delta(p^2 - m^2)}{4m^2} + \delta'(p^2 - m^2)\right]\frac{1}{4}p^\mu\partial^2V(x,p)\\ &\qquad+\mathcal{O}(\mathrm{Kn}^3)\,,\end{aligned} \tag{25}$$

where $\delta'(x) \equiv d\delta(x)/dx = -\delta(x)/x$ is the first order derivative of the delta function. This result agree with Refs. [1, 2] at zeroth and first order in space-time gradient. At second order, we obtain two terms proportional to $\partial^2 V(x,p)$, the momentum p^μ in one term is fixed on the mass-shell by $\delta(p^2-m^2)$ while p^μ in the other term is not. In general we can redefine $V(x,p)$ and the delta-function as

$$\begin{aligned}&\delta(p^2-m^2-\delta m^2)\left[1+\delta V\right]V(x,p)\\ &\equiv \delta(p^2-m^2)V(x,p)\\ &\quad -\left[\frac{\delta(p^2-m^2)}{4m^2}+\delta'(p^2-m^2)\right]\frac{1}{4}\partial^2 V(x,p)\,, \quad (26)\end{aligned}$$

where

$$\begin{aligned}\delta m^2 &\simeq \frac{1}{4V(x,p)}\partial^2 V(x,p)+\mathcal{O}(\mathrm{Kn}^3)\,,\\ \delta V &\simeq -\frac{1}{16m^2 V(x,p)}\partial^2 V(x,p)+\mathcal{O}(\mathrm{Kn}^3)\,.\end{aligned}$$

In this way we can clearly see interpretations of the second order terms: the on-mass-shell part plays as a modification to the distribution function from long-range correlations, while the off-mass-shell part modifies the mass-shell of particles. We note that in this work, interactions between different particles are neglected, but long-range correlations still exist because particles are treated as wave-packets with finite space-time volume. The vector component of the Wigner function in local thermal equilibrium is obtained by substituting V_{eq} and n^μ_{eq} in Eq. (23) into Eq. (25),

$$\begin{aligned}\mathcal{V}^\mu_{\mathrm{eq}} &= \frac{4}{(2\pi)^3}\delta\left(p^2-m^2-\frac{1}{4}\partial^2\right)\\ &\quad\times\left(1-\frac{1}{16m^2}\partial^2\right)\left(1+\frac{1}{16}\omega^{\alpha\beta}\omega_{\alpha\beta}\right)p^\mu\\ &\quad\times\left[\theta(p^0)\exp(-\beta\cdot p+\xi)+\theta(-p^0)\exp(\beta\cdot p-\xi)\right]\\ &\quad+\frac{1}{2(2\pi)^3m^2}\delta(p^2-m^2)\epsilon^{\mu\nu\alpha\beta}p_\alpha\partial_\nu\epsilon_{\beta\rho\sigma\lambda}p^\rho\omega^{\sigma\lambda}\\ &\quad\times\left[\theta(p^0)\exp(-\beta\cdot p+\xi)-\theta(-p^0)\exp(\beta\cdot p-\xi)\right]\\ &\quad+\mathcal{O}(\mathrm{Kn}^3,\,\mathrm{Kn}^2\chi_s,\,\mathrm{Kn}\chi_s^2,\,\chi_s^3)\,, \quad (27)\end{aligned}$$

where the derivative operator acts on locally-defined parameters β^μ, ξ, and $\omega^{\mu\nu}$. Integrating $\mathcal{V}^\mu_{\mathrm{eq}}$ over momentum, we obtain the current density J^μ_{eq}, which is given in Eq. (14) in the main text. The energy-momentum tensor in local equilibrium is derived by substituting $\mathcal{V}^\mu_{\mathrm{eq}}$ in Eq. (27) into Eq. (12) in the main text. In the calculations, we have taken replacements $p^\mu \to -p^\mu$ for anti-particle parts and used the following rank-l moments

$$\begin{aligned}I^{\mu_1\mu_2\cdots\mu_l} &\equiv \frac{8}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)\theta(p^0)\\ &\quad\times p^{\mu_1}p^{\mu_2}\cdots p^{\mu_l}\exp(-\beta\cdot p)\,. \quad (28)\end{aligned}$$

Here $\beta^\mu \equiv \beta u^\mu = u^\mu/T$ is the thermal velocity. Such a moment expansion for distribution function is widely used for deriving hydrodynamics from kinetic theory [3, 4]. The rank-$0,1,2$ moments are given by

$$\begin{aligned}I &= K_0(\beta)\,,\\ I^\mu &= u^\mu K_1(\beta)\,,\\ I^{\mu\nu} &= u^\mu u^\nu K_2(\beta)+\frac{1}{3}\Delta^{\mu\nu}(m^2K_0-K_2)\,, \quad (29)\end{aligned}$$

and the rank-3 moment $I^{\mu\nu\alpha}$ is given in Eq. (17) in the main text. The projection operator is defined as $\Delta^{\mu\nu} \equiv g^{\mu\nu}-u^\mu u^\nu$. Here we have defined K_n as shown in Eq. (15) in the main text.

The recursive relation for K_n can be proved as follows: we first convert the definition of K_n to another equivalent form

$$K_n = \frac{16\pi}{(2\pi\hbar)^3}\int dp\,p^2(p^2+m^2)^{(n-1)/2}e^{-\beta\sqrt{p^2+m^2}}\,. \quad (30)$$

Then we try to calculate

$$\begin{aligned}&K_n-m^2K_{n-2}\\ &=\frac{16\pi}{(2\pi)^3}\int_0^\infty dp(p^2+m^2)^{(n-3)/2}p^4e^{-\beta\sqrt{p^2+m^2}}\\ &=-\frac{16\pi}{\beta(2\pi)^3}\int_0^\infty dp(p^2+m^2)^{(n-2)/2}p^3\frac{d}{dp}e^{-\beta\sqrt{p^2+m^2}}\\ &=\frac{16\pi}{\beta(2\pi)^3}\int_0^\infty dpe^{-\beta\sqrt{p^2+m^2}}\frac{d}{dp}\left[(p^2+m^2)^{(n-2)/2}p^3\right]\,,\end{aligned} \quad (31)$$

where in the last step we have used the method of integrating by parts. Completing the derivative with respect to p, we obtain

$$\begin{aligned}K_n-m^2K_{n-2} &= \frac{16\pi}{\beta(2\pi)^3}\int_0^\infty dpe^{-\beta\sqrt{p^2+m^2}}\\ &\times\left[3p^2(p^2+m^2)^{(n-2)/2}+(n-2)p^4(p^2+m^2)^{(n-4)/2}\right]\,.\end{aligned} \quad (32)$$

We further use $p^2=(p^2+m^2)-m^2$ for the second term and obtain

$$\begin{aligned}K_n-m^2K_{n-2} &= \frac{16\pi}{\beta(2\pi)^3}\int_0^\infty dpe^{-\beta\sqrt{p^2+m^2}}\\ &\quad\times\Big[(n+1)p^2(p^2+m^2)^{(n-2)/2}\\ &\qquad -m^2(n-2)p^2(p^2+m^2)^{(n-4)/2}\Big]\\ &= \frac{n+1}{\beta}K_{n-1}-\frac{(n-2)}{\beta}m^2K_{n-3}\,. \quad (33)\end{aligned}$$

Note that for massless particles, the relations are simplified to $K_n=[n+1)/\beta]K_{n-1}$.

With the help of fluid velocity vector u^μ and the related projection operator $\Delta^{\mu\nu} \equiv g^{\mu\nu}-u^\mu u^\nu$, one can decompose J^μ_{eq} as

$$\begin{aligned}J^\mu_{\mathrm{eq}} &= (u\cdot J_{\mathrm{eq}})u^\mu+\Delta^\mu_\nu J^\nu_{\mathrm{eq}}\\ &\equiv n_{\mathrm{eq}}u^\mu+\delta j^\mu\,. \quad (34)\end{aligned}$$

In analogue to the viscous hydrodynamics, we identify $n_{\rm eq} \equiv u \cdot J_{\rm eq}$ as the particle number density in the co-moving frame of vector u^μ. The remaining part δj^μ is the diffusion current. In this work, δj^μ consists of two parts: one is the contribution of magnetization current, which is of order $\mathcal{O}({\rm Kn}\,\chi_s)$, while the other part contains second order space-time derivative and therefore is $\mathcal{O}({\rm Kn}^2)$.

On the other hand, the energy-momentum tensor in general can be decomposed into a symmetric part $T_S^{\mu\nu}$ and an anti-symmetric part $\delta T_A^{\mu\nu}$,

$$T_{\rm eq}^{\mu\nu} = T_S^{\mu\nu} + \delta T_A^{\mu\nu} \,. \tag{35}$$

From Eq. (14) in the main text, one can easily find that the anti-symmetric part will vanish for the spinless case with $\omega^{\mu\nu} = 0$. Therefore in our power counting scheme, $\delta T_A^{\mu\nu}$ is of order $\mathcal{O}({\rm Kn}\,\chi_s)$. The symmetric part can be further decomposed as

$$T_S^{\mu\nu} = \epsilon_{\rm eq} u^\mu u^\nu - P_{\rm eq}\Delta^{\mu\nu} + \delta T_S^{\mu\nu} \,, \tag{36}$$

in analogue to that in the viscous hydrodynamics. Similar to δj^μ, we also find that $\delta T_S^{\mu\nu}$ consists of two contributions. One of them depend on the spin potential $\omega^{\mu\nu}$ and will vanish when $\omega^{\mu\nu} = 0$. We identify this part as the heat flow and viscous tensor correction induced by the spin polarization. The other contribution arise from long-range correlations, which contains second order derivatives.

C. Spin tensor

The spin tensor is given by the axial-vector component of the Wigner function as shown in Eq. (13) in the main text. Using Eq. (21) and Eq. (6) in the main text, we obtain

$$\begin{aligned}\mathcal{A}^\mu &= {\rm Tr}\left[\gamma^\mu\gamma^5(W_0 + \delta W)\right] \\ &= \delta(p^2 - m^2) m n^\mu(x,p) + \mathcal{O}({\rm Kn}^2\chi_s) \,,\end{aligned} \tag{37}$$

where ${\rm Tr}\left(\gamma^\mu\gamma^5\delta W\right)$ is a term of $\mathcal{O}({\rm Kn}^2\chi_s)$ and thus does not contribute to $\mathcal{A}^\mu$. Using the equilibrium $n_{\rm eq}^\mu$ in Eq. (23), one can derive the spin tensor $S_{\rm eq}^{\lambda,\mu\nu}$ and the result is given in Eq. (21) in the main text.

The spin potential $\omega^{\mu\nu}$ is antisymmetric in its indices $\mu \leftrightarrow \nu$. In general, we can decompose it as

$$\omega^{\mu\nu} = (\omega^{\mu\alpha}u_\alpha)u^\nu - (\omega^{\nu\alpha}u_\alpha)u^\mu + \epsilon^{\mu\nu\alpha\beta}u_\alpha\tilde{\omega}_\beta \,. \tag{38}$$

Such a decomposition is in analogue to decomposing the electromagnetic field tensor in terms of the electric field and the magnetic field. Here we have defined

$$\tilde{\omega}^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu\omega_{\alpha\beta} \,. \tag{39}$$

We now focus on the following term, which appears in $S_{\rm eq}^{\lambda,\mu\nu}$,

$$\begin{aligned}& u^\lambda\omega^{\mu\nu} + 2u^{[\mu}\omega^{\nu]\lambda} \\ &= u^\lambda(\omega^{\mu\alpha}u_\alpha)u^\nu - u^\lambda(\omega^{\nu\alpha}u_\alpha)u^\mu + u^\lambda\epsilon^{\mu\nu\alpha\beta}u_\alpha\tilde{\omega}_\beta \\ &\quad + u^\mu(\omega^{\nu\alpha}u_\alpha)u^\lambda - u^\mu(\omega^{\lambda\alpha}u_\alpha)u^\nu + u^\mu\epsilon^{\nu\lambda\alpha\beta}u_\alpha\tilde{\omega}_\beta \\ &\quad - u^\nu(\omega^{\mu\alpha}u_\alpha)u^\lambda + u^\nu(\omega^{\lambda\alpha}u_\alpha)u^\mu - u^\nu\epsilon^{\mu\lambda\alpha\beta}u_\alpha\tilde{\omega}_\beta \\ &= \left(u^\lambda\epsilon^{\mu\nu\alpha\beta} + u^\mu\epsilon^{\nu\lambda\alpha\beta} - u^\nu\epsilon^{\mu\lambda\alpha\beta}\right)u_\alpha\tilde{\omega}_\beta \,.\end{aligned} \tag{40}$$

We then use the Schouten identity [5],

$$\begin{aligned}& u^\lambda\epsilon^{\mu\nu\alpha\beta} + u^\mu\epsilon^{\nu\alpha\beta\lambda} \\ &\quad + u^\nu\epsilon^{\alpha\beta\lambda\mu} + u^\alpha\epsilon^{\beta\lambda\mu\nu} + u^\beta\epsilon^{\lambda\mu\nu\alpha} = 0 \,,\end{aligned} \tag{41}$$

and obtain

$$\begin{aligned}u^\lambda\omega^{\mu\nu} + 2u^{[\mu}\omega^{\nu]\lambda} &= -\left(u^\beta\epsilon^{\lambda\mu\nu\alpha} + u^\alpha\epsilon^{\beta\lambda\mu\nu}\right)u_\alpha\tilde{\omega}_\beta \\ &= \epsilon^{\lambda\mu\nu\beta}\tilde{\omega}_\beta \,,\end{aligned} \tag{42}$$

where we have used $u^\mu u_\mu = 1$ and $u^\mu\tilde{\omega}_\mu = 0$ in the last step. Substitute the above equation into Eq. (21) in the main text, one can obtain Eq. (22) in the main text.

D. Dipole moment tensor

The dipole moment tensor is given by the tensor component of the Wigner function. The tensor component of W_0 is easy to write according to Eq. (21),

$${\rm Tr}(\sigma^{\mu\nu}W_0) = -\delta(p^2 - m^2)\epsilon^{\mu\nu\alpha\beta}p_\alpha n_\beta \,. \tag{43}$$

On the other hand, the tensor component of δW is obtained by substitute Eq. (21) into Eq. (6) in the main text and then projecting onto $\sigma^{\mu\nu}$,

$$\begin{aligned}{\rm Tr}(\sigma^{\mu\nu}\delta W) = &-\frac{1}{2m}\delta(p^2 - m^2)(p^\mu\partial^\nu - p^\nu\partial^\mu)V \\ &+ \mathcal{O}({\rm Kn}^2\chi_s) \,,\end{aligned} \tag{44}$$

where we also truncate higher order terms. The physical interpretations of Eqs. (43) and (44) can be found in the rest frame with respect to a given p^μ. We identify ${\rm Tr}(\sigma^{\mu\nu}W_0)$ as the contribution form particle's intrinsic magnetic dipole moment, which is related to particle's spin polarization. Meanwhile, ${\rm Tr}(\sigma^{\mu\nu}\delta W)$ is the electric dipole moment contributed by inhomogeneity of particle distribution. The dipole moment tensor in local equilibrium is then given by

$$\begin{aligned}D_{\rm eq}^{\mu\nu}(x) &= \int d^4p\,{\rm Tr}\left[\sigma^{\mu\nu}(W_0 + \delta W)\right] \\ &= \frac{1}{2m}\omega^{\mu\nu}(K_2 - \beta^{-1}K_1)\sinh\xi \\ &\quad + \frac{1}{m}\omega^{\alpha[\mu}u^{\nu]}u_\alpha(K_2 + \beta^{-1}K_1)\sinh\xi \\ &\quad + \frac{1}{m}\partial^{[\mu}\left(u^{\nu]}K_1\sinh\xi\right) + \mathcal{O}({\rm Kn}^2\chi_s) \,,\end{aligned} \tag{45}$$

where we have used the equilibrium expressions of $V_{\rm eq}$ and $n_{\rm eq}^\mu$ in Eq. (23).

III. EOMS FOR THERMODYNAMICAL PARAMETERS

In this section, we will show how to derive the EOMs for thermodynamical parameters $\beta=1/T$, ξ, u^μ, and $\omega^{\mu\nu}$, from conservation laws. Substituting the equilibrium current density $J^\mu_{\rm eq}(x)$ and energy-momentum tensor $T^{\mu\nu}_{\rm eq}(x)$ into the corresponding conservations laws, we derive

$$\begin{aligned}&\partial_\mu(u^\mu K_1\sinh\xi)=0\,,\\&\partial_\mu\left\{\left[u^\mu u^\nu K_2+\frac{1}{3}\Delta^{\mu\nu}(m^2K_0-K_2)\right]\cosh\xi\right\}=0\,.\end{aligned}\tag{46}$$

We note that contributions of $\omega^{\mu\nu}$ vanish in these equations. The EOM for $\omega^{\mu\nu}$ will be derived from the angular momentum conservation in later discussions. Since K_n are pure functions of β, one can find that

$$\partial_\mu K_n=-\left(\partial_\mu\beta\right)K_{n+1}\,.\tag{47}$$

Then conservation laws in (46) give

$$\begin{aligned}&\left(\theta K_1-\dot\beta K_2\right)\sinh\xi+\dot\xi K_1\cosh\xi=0\,,\\&-\left[\frac{\theta}{3}\left(m^2K_0-4K_2\right)+\dot\beta K_3\right]\cosh\xi+\dot\xi K_2\sinh\xi=0\,,\end{aligned}\tag{48}$$

and

$$\begin{aligned}\dot u^\nu\left(m^2K_0-4K_2\right)\cosh\xi-\frac{1}{3}(\nabla^\nu\beta)\left(m^2K_1-K_3\right)\cosh\xi&\\+\frac{1}{3}(\nabla^\nu\xi)\left(m^2K_0-K_2\right)\sinh\xi=0\,.&\end{aligned}\tag{49}$$

Here the first equation in (48) is the charge conservation and the second equation in (48) is obtained from the energy-momentum conservation by projecting onto u^μ, while Eq. (49) is obtained from the energy-momentum conservation by projecting onto the direction perpendicular to u^μ. Here dot represents the covariant time derivative $d/d\tau\equiv u_\mu\partial^\mu$ and $\nabla^\mu\equiv\Delta^{\mu\nu}\partial_\nu$ is the space derivative in comoving frame. By solving Eq. (48), we obtain $\dot\beta$ and $\dot\xi$, while $\dot u^\mu$ is obtained from Eq. (49),

$$\begin{aligned}\dot\beta&=\frac{-K_2\sinh^2\xi-\frac{1}{3}\left(m^2K_0-4K_2\right)\cosh^2\xi}{K_1K_3\cosh^2\xi-K_2K_2\sinh^2\xi}K_1\theta\,,\\\dot\xi&=\frac{-\frac{1}{3}\left(m^2K_0-4K_2\right)K_2-K_1K_3}{K_1K_3\cosh^2\xi-K_2K_2\sinh^2\xi}\theta\sinh\xi\cosh\xi\,,\\\dot u^\mu&=-\frac{m^2K_0-K_2}{3(m^2K_0-4K_2)}\tanh\xi\nabla^\mu\xi\\&\quad+\frac{m^2K_1-K_3}{3(m^2K_0-4K_2)}\nabla^\mu\beta\,.\end{aligned}\tag{50}$$

They can be further simplified using the recursive relation of K_n in Eq. (33). We choose to express K_0 and K_3 in terms of K_1 and K_2 as

$$\begin{aligned}K_0&=\frac{1}{m^2}\left(K_2-\frac{3}{\beta}K_1\right)\,,\\K_3&=\frac{3}{\beta}\left(K_2+\frac{1}{\beta}K_1\right)+m^2K_1\,.\end{aligned}\tag{51}$$

Using these relations to substitute K_0 in $\dot\beta$, $\dot\xi$, and $\dot u^\mu$, as well as K_3 in $\dot u^\mu$, we obtain results shown in Eq. (28) in the main text.

On the other hand, the EOM of $\omega^{\mu\nu}$ is derived by inserting $T^{\mu\nu}_{\rm eq}$ and $S^{\lambda,\mu\nu}_{\rm eq}$ into the angular momentum conservation law, i.e., Eq. (29) in the main text. After a few steps of calculation, we obtain

$$\begin{aligned}m^2K_1\dot\omega^{\mu\nu}\cosh\xi=-m^2\omega^{\mu\nu}\left(\theta+\frac{d}{d\tau}\right)(K_1\cosh\xi)&\\-\partial_{x\lambda}\left[\cosh\xi\left(I^{\mu\lambda\rho}\omega^\nu{}_\rho-I^{\nu\lambda\rho}\omega^\mu{}_\rho\right)\right]\,.&\end{aligned}\tag{52}$$

Since $\omega^{\mu\nu}$ can be decomposed as shown in Eq. (38), we can also decompose $\dot\omega^{\mu\nu}$ in a similar way using the fluid velocity vector u^μ,

$$\dot\omega^{\mu\nu}=\Delta^\mu_\alpha\Delta^\nu_\beta\dot\omega^{\alpha\beta}-u^\mu\dot\omega^{\nu\alpha}u_\alpha+u^\nu\dot\omega^{\mu\alpha}u_\alpha\,.\tag{53}$$

We can solve $\dot\omega^{\mu\nu}u_\nu$ (or $\Delta^\mu_\alpha\Delta^\nu_\beta\dot\omega^{\alpha\beta}$) by contracting Eq. (52) with u_ν (or with projection operators). The results are given in Eqs. (31) and (32) in the main text. In the calculations, we also used the recursive relation of K_n in Eq. (33) to simplify the coefficients. We express all coefficients in terms of K_1 and K_2 because they are respectively related to the particle number density and the energy density as $K_1\simeq n_{\rm eq}/\sinh\xi+\mathcal{O}({\rm Kn}^2,\,{\rm Kn}\,\chi_s,\,\chi_s^2)$ and $K_2\simeq\epsilon_{\rm eq}/\cosh\xi+\mathcal{O}({\rm Kn}^2,\,{\rm Kn}\,\chi_s,\,\chi_s^2)$.

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