Supplementary Materials for "A two-dimensional architecture for fast large-scale trapped-ion quantum computing"

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I. TRANSVERSE DYNAMICS OF A 2D ION CRYSTAL

Suppose N ions are trapped by some external potential in a 2D array with equilibrium positions $(x_i,\,y_i)$ $(i = 1,\,2,\,\cdots,\,N)$. Here we consider the transverse motion of the ions in the z direction with a strong harmonic trapping ω_z. If the displacements of the ions $|z_i|$ are much smaller than their minimal distance d, the (classical) equation of motion is given by

$$m\ddot{z}_i = -m\omega_z^2 z_i + \frac{1}{2}\frac{e^2}{4\pi\epsilon_0}\sum_{j\neq i}\frac{z_i - z_j}{[(x_i-x_j)^2+(y_i-y_j)^2]^{3/2}}, \tag{1}$$

from which we can solve the transverse dynamics. We can regard it as a matrix equation

$$m\frac{d^2\boldsymbol{z}}{dt^2} = -V\boldsymbol{z}, \tag{2}$$

with the potential matrix given by

$$V_{ij} = \begin{cases} \frac{e^2}{4\pi\epsilon_0}\frac{1}{[(x_i-x_j)^2+(y_i-y_j)^2]^{3/2}} & (i\neq j) \\ m\omega_z^2 - \frac{e^2}{4\pi\epsilon_0}\sum_{k\neq i}\frac{1}{[(x_i-x_k)^2+(y_i-y_k)^2]^{3/2}} & (i=j) \end{cases}. \tag{3}$$

We can diagonalize this matrix to solve the collective modes of the ions and their time evolution; or we can quantize these collective modes for designing entangling gates.

In particular, for a square lattice with separation d, we define its lattice vectors

$$\boldsymbol{a}_1 = d\,(1,\,0,\,0)\,, \qquad \boldsymbol{a}_2 = d\,(0,\,1,\,0)\,, \tag{4}$$

with the corresponding reciprocal vectors

$$\boldsymbol{b}_1 = (1,\,0,\,0)\,, \qquad \boldsymbol{b}_2 = (0,\,1,\,0)\,. \tag{5}$$

The position of an ion on the 2D lattice can now be represented by two integer indices α and β as $\boldsymbol{r}_{\alpha\beta} = \alpha\boldsymbol{a}_1 + \beta\boldsymbol{a}_2$. Plugging this expression into the potential matrix, we get

$$V_{\alpha\beta,\alpha'\beta'} = \begin{cases} \frac{e^2}{4\pi\epsilon_0 d^3}\frac{1}{[(\alpha-\alpha')^2+(\beta-\beta')^2]^{3/2}} & (\alpha\neq\alpha' \text{ or } \beta\neq\beta') \\ m\omega_z^2 - \frac{e^2}{4\pi\epsilon_0 d^3}\sum_{(\mu,\lambda)\neq(\alpha,\beta)}\frac{1}{[(\alpha-\mu)^2+(\beta-\lambda)^2]^{3/2}} & (\alpha=\alpha',\,\beta=\beta') \end{cases}. \tag{6}$$

For a finite number of ions, we use the above method to find the collective modes. In the main text we also consider the limit of infinite number of ions. In this case the collective modes in the z direction are travelling waves described by the wave vector $\boldsymbol{k} = k_1\boldsymbol{b}_1 + k_2\boldsymbol{b}_2$ $(k_1, k_2 \in (-\pi/d, \pi/d])$, with a mode vector

$$z^{\boldsymbol{k}}_{\alpha\beta}(t) \propto \exp\left[i(\boldsymbol{k}\cdot\boldsymbol{r}_{\alpha\beta} - \omega_{\boldsymbol{k}}t)\right] = \exp\left[i(\alpha k_1 d + \beta k_2 d - \omega_{\boldsymbol{k}}t)\right]. \tag{7}$$

The mode frequency can be solved by substituting this mode vector into the equation of motion. We get

$$\omega_{\boldsymbol{k}} = \omega_z\sqrt{1 - \epsilon{\sum_{\alpha,\beta}}'\frac{1-\cos(\alpha k_1 d + \beta k_2 d)}{(\alpha^2+\beta^2)^{3/2}}} \tag{8}$$

$$\approx \omega_z\left[1 - \epsilon{\sum_{\alpha,\beta}}'\frac{1-\cos(\alpha k_1 d + \beta k_2 d)}{2(\alpha^2+\beta^2)^{3/2}}\right], \tag{9}$$

where the notation ${\sum}'_{\alpha,\beta}$ means a summation over all integer pairs of α and β apart from when they are both zero.

II. NUMERICAL COMPUTATION OF GATE FIDELITY

Suppose we want to entangle the ions i and j within the N-ion crystal by applying M simultaneous spin-dependent momentum kicks on them with a fixed repetition rate. For the initial state of an ion in $|0\rangle$, suppose the direction of the kicks are $s_1, s_2, \cdots, s_M$ ($s_i = \pm 1$) along the z direction (and for an ion initially in $|1\rangle$ the directions are $-s_1, -s_2, \cdots, -s_M$). Because the evolution of the internal states is completely determined as they are flipped by each SDK, we can focus on the evolution of the motional states. Then the computation of the gate fidelity is very similar to that of a Molmer-Sorensen-like gate under continuous-wave (CW) laser driving (see e.g. Ref. [1]). All what we need is to replace the CW case $\Omega(t)\sin(\mu t+\varphi)$ by a series of delta functions $\sum_{l=1}^{M} s_l \delta(t-t_l)$ where t_l is periodic and denotes the arrival time of the l-th SDK. Following the notation of Ref. [1], we have

$$\alpha_j^k = i\eta_k b_j^k \sum_{l=1}^{M} s_l e^{i\omega_k t_l}, \tag{10}$$

and

$$\Theta_{ij} = -2\sum_k \eta_k^2 b_i^k b_j^k \sum_{l=2}^{M}\sum_{m=1}^{l-1} s_l s_m \sin\omega_k(t_l - t_m). \tag{11}$$

Note that here we have an additional negative sign in Θ_{ij} compared with the definition in Ref. [1] to make it positive. The unitary time evolution is

$$U = \exp\left[-i\Theta_{ij}\hat{\sigma}_z^i\hat{\sigma}_z^j + \sum_k (\alpha_i^k\hat{\sigma}_z^i + \alpha_j^k\hat{\sigma}_z^j)a_k^\dagger - (\alpha_i^{k*}\hat{\sigma}_z^i + \alpha_j^{k*}\hat{\sigma}_z^j)a_k\right]. \tag{12}$$

The gate infidelity for an initial state $|+\rangle|+\rangle$ is then

$$\delta F = \left(\Theta_{ij} - \frac{\pi}{4}\right)^2 + \sum_k \left(|\alpha_i^k|^2 + |\alpha_j^k|^2\right)\coth\frac{\hbar\omega_k}{2k_B T}, \tag{13}$$

where T is the temperature of the initial thermal distribution of the motional state. We can verify that, when restricted to the two-ion case, this result is consistent with Ref. [2] after averaging over the thermal state. As we show in Ref. [1], the average gate infidelity over all initial states is related to this "worst case" infidelity by a factor of 4/5.

III. PARALLEL ENTANGLING GATES

In Ref. [3] we have studied the crosstalk error on a 1D uniform chain and a 2D hexagonal lattice. The crosstalk error of addressing two ions i and j simultaneously is a two-qubit rotation term $\exp(-i\Theta_{ij}\hat{\sigma}_z^i\hat{\sigma}_z^j)$, with Θ_{ij} decaying inverse cubically with the distance between the two ions.

The derivation for 2D square lattice is very similar. The difference from the hexagonal lattice is that, now for the square lattice, the norm of a displacement vector $\boldsymbol{r}_{\alpha\beta} = \alpha\boldsymbol{a}_1 + \beta\boldsymbol{a}_2$ is $d\sqrt{\alpha^2+\beta^2}$ instead of $d\sqrt{\alpha^2+\beta^2+\alpha\beta}$. With this substitution, the rest of the derivation is the same as Appendix A of Ref. [3]. Finally we get a scaling of $1/(n^2+m^2)^{3/2}$ for the two-qubit rotation angle, or $1/(n^2+m^2)^3$ for the crosstalk gate infidelity when parallelizing two entangling gates separated by the displacement vector $\boldsymbol{r}_{nm}$.

What we get above is the crosstalk error for two gates, while in Fig. 4 of the main text we are trying to parallelizing all the entangling gates on a sublattice with separation nd. The infidelity of these two-qubit rotation terms on different ion pairs simply add up together. For each pair of entangling gates to be parallelized, we have four crosstalk terms among the four involved ions. When n is large, we can ignore the small difference in the distance of these four terms (for example, when the four ions are on the same line, the actual distances are $n-1$, n, n and $n+1$). Then we need to evaluate

$$\sum_{\alpha,\beta}{}' \frac{1}{n^6}\frac{1}{(\alpha^2+\beta^2)^3} \approx \frac{4.659}{n^6}. \tag{14}$$

Note that every two entangling gates have four ion pairs for the crosstalk error, and that such crosstalk errors are shared by these two gates. Also note that at $n=1$ the "crosstalk error" is just the entangling gate we want to realize with $\Theta_{ij} = \pi/4$. We finally get the crosstalk error per entangling gate as

$$\left(\frac{\pi}{4}\right)^2 \times 4 \times \frac{1}{2} \times \frac{4.659}{n^6} \approx \frac{5.75}{n^6}, \tag{15}$$

which is valid for large n.

IV. PROPAGATION OF LOCAL DISTURBANCE

Consider an infinite square lattice. According to Sec. I, the transverse modes are travelling waves with frequencies distributed in a narrow band of $O(\epsilon\omega_z)$. We can thus use the group velocity to characterize the speed of propagation

$$\boldsymbol{v}_g(\boldsymbol{k}) = \nabla_{\boldsymbol{k}}\omega(\boldsymbol{k}) \approx -\frac{\epsilon\omega_z d}{2}{\sum_{\alpha,\beta}}' \frac{\sin(\alpha k_1 d + \beta k_2 d)}{(\alpha^2+\beta^2)^{3/2}}\,(\alpha,\,\beta)\,, \tag{16}$$

where we use the fact $\epsilon \ll 1$. In Fig. 1 we numerically evaluate Eq. (16) for each $\boldsymbol{k}$ on a 201×201 square lattice as the blue dots. They are bounded by a maximal group velocity of about $3.5\epsilon\omega_z d$.

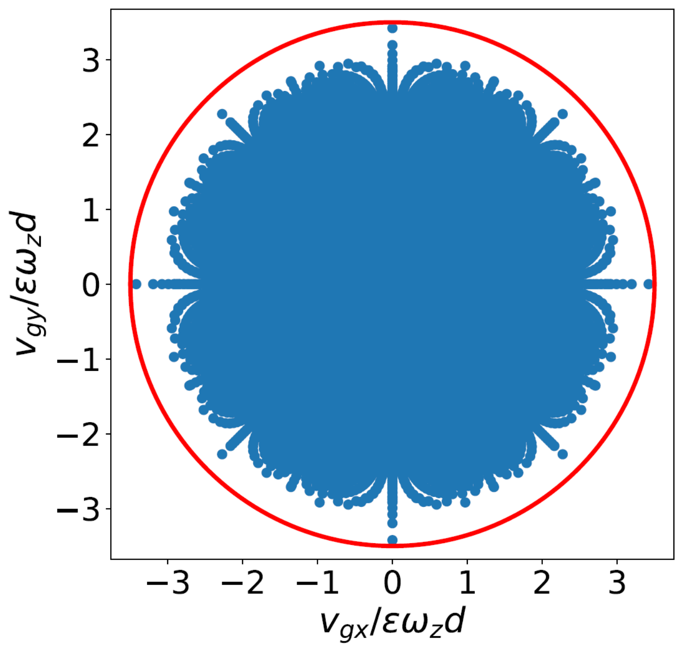

FIG. 1. Group velocity in a 201×201 square lattice with $\epsilon \ll 1$. The red circle indicates a maximal group velocity of about $v_g \approx 3.5\epsilon\omega_z d$.

Strictly speaking, the speed of signal in this system is the speed of light c, which we take as infinity when writing down an electrostatic Coulomb potential. Here we further study the response of far-away ions outside the "light cone" defined by v_g.

Suppose we have an initial disturbance at $t = 0$ on the central ion to give it a displacement z_0 and a velocity v_0, while all the other ions stay at rest. Having solved the collective modes, we can express the response of other ions as

$$\begin{aligned} z_{nm}(t) =& \mathrm{Re}\frac{d^2}{4\pi^2}\int_{-\pi/d}^{\pi/d} dk_1 \int_{-\pi/d}^{\pi/d} dk_2 \left[z_0 + \frac{iv_0}{\omega(\boldsymbol{k})}\right] e^{i[nk_1 d + mk_2 d - \omega(\boldsymbol{k})t]} \\ \approx& \mathrm{Re}\frac{d^2}{4\pi^2}\left[z_0 + \frac{iv_0}{\omega_z}\right]\int_{-\pi/d}^{\pi/d} dk_1 \int_{-\pi/d}^{\pi/d} dk_2 e^{i[nk_1 d + mk_2 d - \omega(\boldsymbol{k})t]} \end{aligned} \tag{17}$$

where we use the fact that $\omega(\boldsymbol{k}) \approx \omega_z$ for $\epsilon \ll 1$ and only keep the $\boldsymbol{k}$ dependence in the phase factor.

Now we study the scaling of the above expression vs. n and m. Similar to our calculation for parallel gates and the derivations in Appendix A of Ref. [3], with some algebra, we can decompose the above integral into several terms like

$$\sin\{\omega_z t[1-\epsilon\zeta]\}\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy \cos[\lambda S(x,y)]\cos(nx+my) \tag{18}$$

where $\lambda \equiv \epsilon\omega_z t$, $S(x,y) \equiv \sum'_{\alpha,\beta}\cos(\alpha x + \beta y)/2(\alpha^2+\beta^2)^{3/2}$, $\zeta \equiv S(0,0)$ and $x = k_1 d$, $y = k_2 d$. In this expression the first term corresponds to a fast oscillation at the local trap frequency; while the terms in the integral describe the slow change of the pulse shape as it propagates.

From Ref. [3] and Sec. III we know that this integral scales as $1/(n^2+m^2)^{3/2}$ for large n and m. With similar derivations, we can bound the other terms appearing in the original integral in Eq. (17) and finally the amplitude is also bounded by $1/(n^2+m^2)^{3/2}$. Furthermore, this scaling shall become valid when n or m is much larger than $\lambda = \epsilon\omega_z t$, which is the only other scale defined in the above expression. In other words, this $1/(n^2+m^2)^{3/2}$ scaling is valid on the sites where the "light cone" defined by the group velocity has not reached.

Therefore we expect a response of $(v_g t/d)(n^2+m^2)^{-3/2}$ for small t. Then the total response of other ions at a gate time of $O(1/\omega_z)$ is $O(\epsilon)$. This explains the good performance of our gate scheme in the multi-ion case.

V. FURTHER NUMERICAL EXAMPLES

Here we consider $^9\mathrm{Be}^+$ and $^{40}\mathrm{Ca}^+$ as two further numerical examples. Since current experiments do not use pulsed laser for these systems, here we assume a laser frequency red-detuned to the $D1$ transition of the ion with a repetition rate $\omega_{\mathrm{rep}} = 2\pi\times 80\,\mathrm{MHz}$. Also we use $\Gamma = 2\pi\times 20\,\mathrm{MHz}$ to estimate the Doppler temperature. For $^9\mathrm{Be}^+$ we consider a laser wavelength of 318 nm and get (1) $d = 50\,\mu\mathrm{m}$, $M = 43$, $\omega_x = 2\pi\times 1.861\,\mathrm{MHz}$, $\epsilon = 0.0009$, $T = 1.075\,\mu\mathrm{s}$, $F_{10\times10} = 0.9994$; and (2) $d = 250\,\mu\mathrm{m}$, $M = 114$, $\omega_x = 2\pi\times 0.7018\,\mathrm{MHz}$, $\epsilon = 5.1\times 10^{-5}$, $T = 2.85\,\mu\mathrm{s}$, $F_{10\times10} = 1 - 1.9\times 10^{-5}$. For $^{40}\mathrm{Ca}^+$ we use a wavelength of 400 nm and obtain (1) $d = 50\,\mu\mathrm{m}$, $M = 86$, $\omega_x = 2\pi\times 0.9306\,\mathrm{MHz}$, $\epsilon = 0.00081$, $T = 2.15\,\mu\mathrm{s}$, $F_{10\times10} = 0.9998$; and (2) $d = 250\,\mu\mathrm{m}$, $M = 227$, $\omega_x = 2\pi\times 0.3524\,\mathrm{MHz}$, $\epsilon = 4.5\times 10^{-5}$, $T = 5.675\,\mu\mathrm{s}$, $F_{10\times10} = 1 - 4\times 10^{-5}$.

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