Supplemental Material for Universal minimum conductivity in disordered double-Weyl Semimetal

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In this Supplemental Material, we present the detailed calculation of Born approximation in section I. The derivation of conductivity and discuss the vertex correction in section II.

I. BORN APPROXIMATION

Let us start from solving the continuous model Hamiltonian Eq.(1) in the main text. We consider the coordinate transformation $k_x = \sqrt{\frac{k\sin\theta}{v_\perp}}\cos\phi$, $k_y = \sqrt{\frac{k\sin\theta}{v_\perp}}\sin\phi$, $k_z = \frac{k}{v_z}\cos\theta$.

$$\begin{aligned} H &= v_\perp(k_x^2 - k_y^2)\sigma_x + v_\perp 2k_x k_y\sigma_y + v_z k_z\sigma_z. \\ &= k\begin{pmatrix} \cos\theta & \sin\theta e^{-i2\phi} \\ \sin\theta e^{i2\phi} & -\cos\theta \end{pmatrix} \end{aligned} \tag{S1}$$

The energy dispersion is $\epsilon_{k,\pm} = \pm k = \pm\sqrt{v_\perp(k_x^2+k_y^2)^2 + v_z k_z^2}$, and the corresponding eigenvector is,

$$\begin{aligned} |+\rangle &= \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i2\phi} \end{pmatrix}, |-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i2\phi} \end{pmatrix}, \\ U &= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i2\phi} & \cos\frac{\theta}{2}e^{i2\phi} \end{pmatrix}, UU^\dagger = I. \end{aligned} \tag{S2}$$

The disorder averaged Green's function is $\hat{G} = \hat{G}_0 + \hat{G}_0\hat{\Sigma}\hat{G}$, the self energy matrix from the Born approximation can be written as $\hat{\Sigma}(E) = \overline{\hat{V}\hat{G}(E)\hat{V}}$, where the disorder matrix is $\hat{V} = \sum_{\mathbf{r}} V(\mathbf{r})|\mathbf{r}\rangle\sigma_0\langle\mathbf{r}|$. Using the transformation matrix U,

$$\begin{aligned} \langle k,s|\hat{V}|p,\nu\rangle &= \frac{1}{N}\sum_r V(\mathbf{r})e^{i(\mathbf{k}-\mathbf{p})\cdot\mathbf{r}}\begin{pmatrix} V_{++} & V_{+-} \\ V_{-+} & V_{--} \end{pmatrix}. \\ V_{++} &= \cos(\frac{\theta_k}{2})\cos(\frac{\theta_p}{2}) + e^{-i2(\phi_k-\phi_p)}\sin(\frac{\theta_k}{2})\sin(\frac{\theta_p}{2}), \\ V_{+-} &= -\cos(\frac{\theta_k}{2})\sin(\frac{\theta_p}{2}) + e^{-i2(\phi_k-\phi_p)}\sin(\frac{\theta_k}{2})\cos(\frac{\theta_p}{2}), \\ V_{-+} &= -\sin(\frac{\theta_k}{2})\cos(\frac{\theta_p}{2}) + e^{-i2(\phi_k-\phi_p)}\cos(\frac{\theta_k}{2})\sin(\frac{\theta_p}{2}), \\ V_{--} &= \sin(\frac{\theta_k}{2})\sin(\frac{\theta_p}{2}) + e^{-i2(\phi_k-\phi_p)}\cos(\frac{\theta_k}{2})\cos(\frac{\theta_p}{2}). \end{aligned} \tag{S3}$$

And the Green's function becomes diagonal under the transformation,

$$\begin{aligned} \langle p,\nu|\hat{G}|p',\nu'\rangle &= \begin{pmatrix} g_+(p,E) & 0 \\ 0 & g_-(p,E) \end{pmatrix}\delta_{pp'}, \\ g_\pm(p,E) &= \frac{1}{E\mp\epsilon_p + i\eta} \end{aligned} \tag{S4}$$

Now we can express the self energy matrix under the eigenspace as,

$$\begin{aligned} &\langle k,s|\hat{\Sigma}|k',s'\rangle = \hat{\Sigma}_{ks,k's'}(E) = \overline{\langle k,s|\hat{V}\hat{G}(E)\hat{V}|k',s'\rangle} \rightarrow \begin{pmatrix} \Sigma_{++} & \Sigma_{+-} \\ \Sigma_{-+} & \Sigma_{--} \end{pmatrix}\delta_{kk'} \\ &\Sigma_{++} = \frac{1}{N}\sum_p\frac{W^2}{12}\frac{1}{2}(g_+ + g_-) \\ &\Sigma_{--} = \Sigma_{++}. \\ &\Sigma_{+-} = 0, \Sigma_{-+} = 0 \end{aligned} \tag{S5}$$

After performing the angle integration, the off diagonal term Σ_{+-} and Σ_{-+} vanish, and the diagonal term become identical $\Sigma_{+-} = \Sigma_{-+} = \Sigma(E)$ which leads to $\hat{\Sigma}_{ks,k's'}(E) = \overline{\langle k,s|\hat{V}\hat{G}(E)\hat{V}|k',s'\rangle} = \Sigma(E)\delta_{kk'}\delta_{ss'}$. The calculation shows that the random scalar disorder do not generate off-diagonal term in self energy matrix and the self energy is independent on the band index ss' and input momentum k. Then, we calculate the momentum space integration to obtain explicit expression.

$$\begin{aligned} \Sigma(E) &= \frac{1}{N}\sum_p\frac{W^2}{12}\frac{1}{2}(g_+ + g_-) \\ &= \frac{a_L^3W^2}{12}\frac{2\pi^2}{(2\pi)^3}\frac{1}{2v_\perp v_z}\int_0^{\omega_c} kdk\frac{(E+i\eta)}{(E+i\eta)^2 - k^2}, \\ &= -\gamma E\ln\frac{\omega_c}{|E|} - i\frac{\pi}{2}\gamma E \end{aligned} \tag{S6}$$

where the last line is obtained by taking the limit $\eta \to 0$, we have defined $\gamma = \frac{a_L^3W^2}{96\pi v_\perp v_z}$. These results is consistent with the numerical simulation for weak disorder. From the real part of self energy, we obtain the quasi-particle residue scale as $Z_E \propto (\ln\frac{\omega_c}{|E|})^{-1}$ as $E \to 0$, the appearance of logarithmic singularities signify the strong modification of quasiparticle properties in the low energy regime. After considering all multiscattering effects, the results of Born approximation are replaced by a more general power law function as discussed in the main text.

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II. KUBO-GREENWOOD FORMULA

$$\sigma_{\mu\mu}(\mathrm{E_F}) = e^2 \int d\omega(-\frac{\partial f(\omega, E_F)}{\partial\omega}) \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathrm{Tr}(\tilde{v}_\mu A(\mathbf{k}) v_\mu A(\mathbf{k}))$$
$$= -\frac{e^2}{4}\sum_{tt'} tt' \int d\omega(-\frac{\partial f(\omega, E_F)}{\partial\omega}) \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathrm{Tr}[\tilde{v}_\mu \hat{G}^t(\mathbf{k}) v_\mu \hat{G}^{t'}(\mathbf{k})]. \tag{S7}$$

The spectral function is defined as $\hat{A} = (\hat{G}^R - \hat{G}^A)/(2i)$. The bare velocity operator is defined as $v_\mu(\mathbf{k}) = \partial_\mu H(\mathbf{k})$,

$$\begin{aligned} v_x &= 2v_\perp(k_x\sigma_y + k_y\sigma_x) \\ v_y &= 2v_\perp(k_x\sigma_x - k_y\sigma_y) \\ v_z &= v_z\sigma_z \end{aligned} \tag{S8}$$

By directly calculating the trace using the eigen vector Eq.(S2) and complete the integration, we obtain the Eq.(7) in the main text. In Fig .S1, we compare the Zero temperature conductivity calculated from the data points of self energy with the results obtained from the fitting function of self energy in the main text.

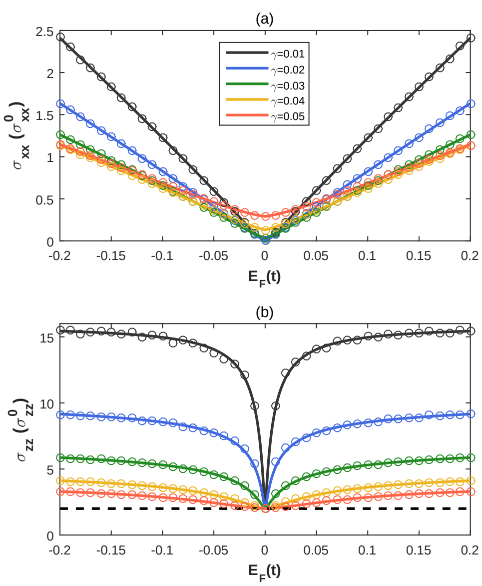

FIG. S1. Zero temperature conductivity along k_x direction σ_{xx} (a) and k_y direction σ_{yy} (b). The open circles are the numerical results calculated from the data points of self energy, the solid lines are obtained from the fitting function of self energy.

The vertex correction to the velocity operator is determined by the Bethe-Salpeter equation (see Fig. S2),

$$\tilde{v}_\mu = v_\mu + < \hat{V}\hat{G}^t \tilde{v}_\mu \hat{G}^{t'} \hat{V} > \tag{S9}$$

We note that there is no vertex correction for the velocity operator $\tilde{v}_\mu(k)$ along the $\hat{x}$ or $\hat{y}$ direction, since the Green's functions are even function of momentum k, while the velocity is

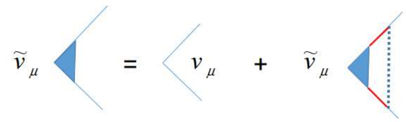

FIG. S2. Bethe-Salpeter equation of velocity operator under the ladder approximation, where v_μ is the bare operator, and the red solid line is disorder averaging Green's function.

odd in momentum, $v_{x/y}(-k) = -v_{x/y}(k)$, and the integration vanishes.

In the $\hat{z}$ direction, the vertex correction from Eq.(S9) can be solved by ladder approximation. Here we consider the solution $\tilde{v}_z = L_{tt'}(E)\sigma_z$, and we have,

$$L_{tt'}(E_F) = \frac{1}{1 - \frac{\gamma}{2}\Pi_{tt'}(E_F)} \tag{S10}$$

with the function $\Pi_{tt'}(E_F)$ defined as,

$$\Pi_{tt'}(E_F) = \int_0^{\omega_c\to\infty} dk \frac{2k(\alpha + it\eta)(E + it'\eta)}{[(\alpha + it\eta)^2 - k^2][(\alpha + it'\eta)^2 - k^2]}$$
$$= \begin{cases} -1, & (t' = t) \\ (\frac{\alpha}{\eta} + \frac{\eta}{\alpha})\tan^{-1}(\frac{\alpha}{\eta}), & (t' \neq t) \end{cases} \tag{S11}$$

Using these results, the conductivity along the $\hat{z}$ direction can be written as,

$$\sigma_{zz}(\mathrm{E_F}) = -\frac{e^2}{4}\sum_{tt'}\int \frac{d^3k}{(2\pi)^3} L_{tt'}(E_F)\mathrm{Tr}(\sigma_z \hat{G}^t(k,\omega) v_\mu \hat{G}^{t'}(k,\omega))$$
$$= -\sigma^0_{zz}\sum_{tt'} tt' L_{tt'}(E_F)\Pi_{tt'}(E_F)$$
$$= \sigma^0_{zz}\frac{1 + (\frac{\alpha}{\eta} + \frac{\eta}{\alpha})\tan^{-1}(\frac{\alpha}{\eta})}{(1 + \frac{\gamma}{2})[1 - \frac{\gamma}{2}(\frac{\alpha}{\eta} + \frac{\eta}{\alpha})\tan^{-1}(\frac{\alpha}{\eta})]} \tag{S12}$$

In the Boltzmann limit $E_F\tau \gg 1$, the function $\Pi_{+-}(E) = (\frac{\alpha}{\eta} + \frac{\eta}{\alpha})\tan^{-1}(\frac{\alpha}{\eta}) \approx \frac{\alpha}{\eta}\frac{\pi}{2}$, if we take the first order Born approximation for $\eta = \frac{1}{2\tau} = \pi\frac{\gamma}{2}E_F$, and drop the real part of self energy ($\alpha = E_F$), we obtain $\Pi_{+-}(E) = \frac{1}{\gamma}$. Thus, we find $\sigma_{zz}(E_F) = 2 \times \sigma^0_{zz}\frac{\alpha}{\eta}\frac{\pi}{2}$ which is twice as large as the value without the vertex correction. At the gapless point $E = 0$, $\Pi_{+-}(E = 0) = 1$, and we obtain,

$$\sigma_{zz}(\mathrm{E_F}) = \frac{2\sigma^0_{zz}}{1 - (\frac{\gamma}{2})^2} \tag{S13}$$

With vertex correction, the minimum conductivity is dependent on strength of disorder. However, this dependence is extremely small (order of γ^2) for weak disorder.