Supplementary Material: Quantization Scheme of Surface Plasmon Polaritons in Two-Dimensional Helical Liquids

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I. HAMILTONIAN IN MOMENTUM SPACE

In this section, let's derive the momentum space Hamiltonian in Heisenberg picture as the start point. In real space, the Hamiltonian reads

$$\hat{\mathbf{H}}(t) = \int d\mathbf{x}\hat{\Psi}^\dagger(\mathbf{x},t)H_s[\hat{\mathbf{p}} + \frac{e}{c}\hat{\mathbf{A}}(\mathbf{x},t)]\hat{\Psi}(\mathbf{x},t) + \frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\hat{\Psi}^\dagger(\mathbf{x},t)\hat{\Psi}^\dagger(\mathbf{x}',t')V(|\mathbf{x}-\mathbf{x}'|)\hat{\Psi}(\mathbf{x}',t)\hat{\Psi}(\mathbf{x},t) + \sum_{\lambda\mathbf{p}}\hbar\omega_{\mathbf{p}}\hat{a}^\dagger_{\lambda\mathbf{p}}(t)\hat{a}_{\lambda\mathbf{p}}(t), \tag{1}$$

where $H_s(\hat{\mathbf{p}})$ is the 2D single electron Hamiltonian

$$H_s(\hat{\mathbf{p}}) = \frac{\hat{\mathbf{p}}^2}{2m^*} + \hbar v_f(\hat{\sigma}_x\hat{p}_y - \hat{\sigma}_y\hat{p}_x) + h\sigma_z - \mu. \tag{2}$$

The real space Hamiltonian Eq. (1) contains four parts

$$\hat{\mathbf{H}}(t) = \hat{\mathbf{H}}_0(t) + \hat{\mathbf{H}}_{fp}(t) + \hat{\mathbf{H}}^p(t) + \hat{\mathbf{H}}^d(t), \tag{3}$$

$\hat{\mathbf{H}}_0(t)$ is the Hamiltonian without magnetic field $\hat{\mathbf{A}}(\mathbf{x},t)$, $\hat{\mathbf{H}}_{fp}(t)$ is the Hamiltonian for free photons,

$$\begin{aligned}\hat{\mathbf{H}}_0(t) &= \int d\mathbf{x}\hat{\Psi}^\dagger(\mathbf{x},t)H_s(\hat{\mathbf{p}})\hat{\Psi}(\mathbf{x},t) + \frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\hat{\Psi}^\dagger(\mathbf{x},t)\hat{\Psi}(\mathbf{x},t)V(|\mathbf{x}-\mathbf{x}'|)\hat{\Psi}^\dagger(\mathbf{x}',t)\hat{\Psi}(\mathbf{x}',t)\\ \hat{\mathbf{H}}_{fp}(t) &= \sum_{\lambda\mathbf{p}}\hbar\omega_{\mathbf{p}}\hat{a}^\dagger_{\lambda\mathbf{p}}(t)\hat{a}_{\lambda\mathbf{p}}(t)\end{aligned} \tag{4}$$

and $\hat{\mathbf{H}}^p(t)$ and $\hat{\mathbf{H}}^d(t)$ are electron photon interacting Hamiltonian

$$\begin{aligned}\hat{\mathbf{H}}^p(t) &= -\frac{1}{c}\int \hat{\mathbf{j}}^p(\mathbf{x},t)\cdot\hat{\mathbf{A}}(\mathbf{x},t)d\mathbf{x},\\ \hat{\mathbf{H}}^d(t) &= -\frac{1}{2c}\int \hat{\mathbf{j}}^d(\mathbf{x},t)\cdot\hat{\mathbf{A}}(\mathbf{x},t)d\mathbf{x}\end{aligned} \tag{5}$$

the paramagnetic and diamagnetic currents can be expressed as

$$\begin{aligned}\hat{\mathbf{j}}^p(\mathbf{x},t) &= \hat{\Psi}^\dagger(\mathbf{x},t)[\frac{ie\hbar}{2m^*}(\nabla - \acute{\nabla}) + e\hbar v_f(\sigma_y\mathbf{i} - \sigma_x\mathbf{j})]\hat{\Psi}(\mathbf{x},t)\\ \hat{\mathbf{j}}^d(\mathbf{x},t) &= -\frac{e^2}{m^*c}\hat{\mathbf{A}}(\mathbf{x},t)\end{aligned} \tag{6}$$

where $\hat{\Psi}^\dagger(\mathbf{x},t)\acute{\nabla}\hat{\Psi}(\mathbf{x},t)$ means $[\nabla\hat{\Psi}^\dagger(\mathbf{x},t)]\hat{\Psi}(\mathbf{x},t)$, and $\mathbf{i}$, $\mathbf{j}$ are unit vectors in x and y direction.

We can expand the operators in momentum space,

$$\begin{aligned}\hat{\Psi}(\mathbf{x},t) &= \frac{1}{\sqrt{S}}\psi(z)\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\Psi}_{\mathbf{k}}(t),\\ \frac{1}{c}\hat{A}(\mathbf{x},t) &= \frac{1}{\sqrt{\nu}}\sum_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}\sqrt{\frac{2\pi\hbar}{\omega_{\mathbf{p}}}}[\xi_\lambda(\mathbf{p})\hat{a}_{\lambda\mathbf{p}}(t) + \xi_\lambda(-\mathbf{p})\hat{a}^\dagger_{\lambda,-\mathbf{p}}(t)],\end{aligned} \tag{7}$$

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where $\omega_{\mathbf{p}} = cp$ is the free photon's frequency. Because we only consider the situations for the chemical potential not very high, we can safely ignore the topological trivial term $\hbar^2k^2/2m^*$ for electrons (quite small compared to the spin-orbit coupling terms). Then the Hamiltonian without magnetic field

$$\hat{\mathbf{H}}_0(t) = \sum_{\mathbf{k}} \hat{\Psi}^\dagger_{\mathbf{k}}[\hbar v_f(\sigma_x k_y - \sigma_y k_x) + h\sigma_z - \mu]\hat{\Psi}_{\mathbf{k}} + \frac{1}{2S}\sum_{\mathbf{kk'q}} V_q \hat{\Psi}^\dagger_{\mathbf{k+q}}\hat{\Psi}^\dagger_{\mathbf{k'-q}}\hat{\Psi}_{\mathbf{k'}}\hat{\Psi}_{\mathbf{k}} \tag{8}$$

Note that $\hbar p/m^* << \hbar k_f/m^* << v_f$ (p is the momentum of vector field), which means we can only keep the spin-orbit coupling terms in $\hat{\mathbf{j}}^p_x(\mathbf{x},t)$, $\hat{\mathbf{j}}^p_y(\mathbf{x},t)$. Together with the fact that $\psi(z)$ is well localized around $z = 0$, and

$$\int d\mathbf{x}\hat{j}_z(\mathbf{x},t)\hat{A}^z(\mathbf{x},t) \approx -\frac{ie\hbar}{m^*c}\int [\hat{A}^z(\mathbf{x},t)]_{z=0}d\mathbf{r}\int \hat{\Psi}^\dagger(\mathbf{x},t)\partial_z\hat{\Psi}(\mathbf{x},t)dz = 0 \tag{9}$$

We have

$$\hat{\mathbf{H}}^p(t) \approx \frac{ev_f}{\sqrt{\nu}}\sum_{\mathbf{kq}p_z}\hat{\Psi}^\dagger_{\mathbf{k+q}}[\sigma_x\hat{A}^y_{(\mathbf{q},p_z)} - \sigma_y\hat{A}^x_{(\mathbf{q},p_z)}]\hat{\Psi}_{\mathbf{k}} \tag{10}$$

The diamagnetic term

$$\begin{aligned}\hat{\mathbf{H}}^d(t) &= \sum_{l=x,y,z}\frac{e^2}{2m^*c^2}\int d\mathbf{r}dz|\psi(z)|^2\hat{n}(\mathbf{r},t)\hat{A}^l(\mathbf{x},t)\hat{A}^l(\mathbf{x},t) \approx \sum_{l=x,y,z}\frac{\bar{n}e^2}{2m^*c^2}\int d\mathbf{r}[\hat{A}^l(\mathbf{x},t)\hat{A}^l(\mathbf{x},t)]|_{z=0} \\ &= \frac{\bar{n}e^2}{2m^*L_z}\sum_{\mathbf{q},p_z,p'_z,l=x,y,z}\hat{A}^l_{(\mathbf{q},p_z)}\hat{A}^l_{(-\mathbf{q},-p'_z)}.\end{aligned} \tag{11}$$

Finally, the total electron light interaction Hamiltonian now reads (from now on we omit t in the $\mathbf{k}$ space operators)

$$\begin{aligned}\hat{H} =& \sum_{\mathbf{k}}\hat{\Psi}^\dagger_{\mathbf{k}}[\hbar v_f(\sigma_x k_y - \sigma_y k_x) + h\sigma_z - \mu]\hat{\Psi}_{\mathbf{k}} + \frac{1}{2S}\sum_{\mathbf{kk'q}}V_q\hat{\Psi}^\dagger_{\mathbf{k+q}}\hat{\Psi}^\dagger_{\mathbf{k'-q}}\hat{\Psi}_{\mathbf{k'}}\hat{\Psi}_{\mathbf{k}} + \frac{ev_f}{\sqrt{\nu}}\sum_{\mathbf{kq}p_z}\hat{\Psi}^\dagger_{\mathbf{k+q}}[\sigma_x\hat{A}^y_{(\mathbf{q},p_z)} - \sigma_y\hat{A}^x_{(\mathbf{q},p_z)}]\hat{\Psi}_{\mathbf{k}} \\ &+ \sum_{\lambda\mathbf{p}}\hbar\omega_{\lambda\mathbf{p}}\hat{a}^\dagger_{\lambda\mathbf{p}}\hat{a}_{\lambda\mathbf{p}} + \frac{\Pi^d}{2L_z}\sum_{\mathbf{q},p_z,p'_z,l=x,y,z}\hat{A}^l_{(\mathbf{q},p_z)}\hat{A}^l_{(-\mathbf{q},-p'_z)}\end{aligned} \tag{12}$$

where $\Pi_d = \frac{\bar{n}e^2}{m^*}$.

It is well known that the eigen functions of $H_{\mathbf{k}} = \frac{\hbar^2k^2}{2m^*} + \hbar v_f(\sigma_x k_y - \sigma_y k_x) + h\sigma_z - \mu$ read

$$|s\mathbf{k}\rangle = e^{\frac{i}{2}s\theta_{\mathbf{k}}}\begin{pmatrix} u_{sk}e^{-\frac{i}{2}\theta_{\mathbf{k}}} \\ -isv_{sk}e^{\frac{i}{2}\theta_{\mathbf{k}}}\end{pmatrix} \tag{13}$$

corresponding to eigen energies $\xi_{sk} = -\mu + sE_k$, where $u_{sk}, v_{sk} = \sqrt{\frac{1}{2}(1 \pm sh/E_k)}$, $s = \pm 1$, $E_k = \sqrt{\hbar^2v_f^2k^2 + h^2}$.

Now, by the following operator transformation $\hat{\gamma}_{s\mathbf{k}} = \langle s\mathbf{k}|\hat{\Psi}_{\mathbf{k}}$, we have

$$\hat{\mathbf{H}} = \sum_{s\mathbf{k}}\xi_{s\mathbf{k}}\hat{\gamma}^\dagger_{s\mathbf{k}}\hat{\gamma}_{s\mathbf{k}} + \sum_{\lambda\mathbf{q}}\hbar\omega_{\mathbf{q}}\hat{a}^\dagger_{\lambda\mathbf{q}}\hat{a}_{\lambda\mathbf{q}} - \frac{1}{\sqrt{\nu}}\sum_{\mathbf{q}}\hat{\mathbf{j}}_{\mathbf{q}}\cdot\hat{\mathbf{A}}_{-\mathbf{q}} + \frac{1}{2S}\sum_{ss'll'\mathbf{kk'q}}V^{ss'll'}_{\mathbf{kq}}\hat{\gamma}^\dagger_{s\mathbf{k+q}}\hat{\gamma}^\dagger_{l\mathbf{k'-q}}\hat{\gamma}_{l'\mathbf{k'}}\hat{\gamma}_{s'\mathbf{k}} \tag{14}$$

where

$$V^{ss'll'}_{\mathbf{kq}} = V_q\langle s\mathbf{k+q}|s'\mathbf{k}\rangle\langle l\mathbf{k'-q}|l'\mathbf{k'}\rangle, \tag{15}$$

and $V_q = 2\pi e^2/q$ is the two dimensional Coulomb potential. The current operator $\hat{\mathbf{j}}(\mathbf{q}) = \hat{\mathbf{j}}^p(\mathbf{q}) + \hat{\mathbf{j}}^d(\mathbf{q})$ contains two parts, and

$$\hat{j}^p(\mathbf{q}) = \sum_{\mathbf{k}}\langle s\mathbf{k+q}|J^x_{\mathbf{kq}}\mathbf{i} + J^y_{\mathbf{kq}}\mathbf{j}|s'\mathbf{k}\rangle = e\hbar v_f\sum_{\mathbf{k}}\langle s\mathbf{k+q}|\sigma_y\mathbf{i} - \sigma_x\mathbf{j}|s'\mathbf{k}\rangle \tag{16}$$

The diamagnetic current arising from the topological trivial term in $\hat{H}_s$ mentioned previously reads

$$\hat{\mathbf{j}}^d_y(\mathbf{q}) = \frac{1}{2}\sqrt{\frac{S}{L_z}}\Pi^d\sum_{p'_z}\hat{A}_{\mathbf{q},p_z+p'_z} \tag{17}$$

II. OPERATOR DYNAMIC EQUATIONS

In this section, we will calculate the operator dynamic equations with random phase approximation (RPA). Let's firstly define the electron density oscillation operator $\hat{\rho}^{\dagger}_{ss'\mathbf{kq}} = \hat{\gamma}^{\dagger}_{s,\mathbf{k+q}}\hat{\gamma}_{s',\mathbf{k}}$, note that $\hat{\rho}_{ss'\mathbf{kq}} = \hat{\rho}^{\dagger}_{s's\mathbf{k+q},-\mathbf{q}}$ and we have the following communication relation

$$[\hat{\rho}_{ss'\mathbf{kq}},\ \hat{\rho}^{\dagger}_{ll'\mathbf{k'q'}}] = \hat{\gamma}^{\dagger}_{s',\mathbf{k}}\hat{\gamma}_{l',\mathbf{k'}}\delta_{sl}\delta_{\mathbf{k+q},\mathbf{k'+q'}} - \hat{\gamma}^{\dagger}_{l,\mathbf{k'+q'}}\hat{\gamma}_{s,\mathbf{k+q}}\delta_{s'l'}\delta_{\mathbf{k},\mathbf{k'}} \tag{18}$$

If we replace the operators in the right hand by their average values when $\mathbf{q},\mathbf{q'} \neq 0$ (RPA), we can find

$$[\hat{\rho}_{ss'\mathbf{kq}},\ \hat{\rho}^{\dagger}_{ll'\mathbf{k'q'}}] = \Delta n^{ss'}_{\mathbf{kq}}\delta_{sl}\delta_{s'l'}\delta_{\mathbf{kk'}}\delta_{\mathbf{qq'}}, \tag{19}$$

and the time evolution $\partial_t\hat{O} = i/\hbar[\hat{H},\hat{O}]$ for electrons reads

$$\begin{aligned}
\partial_t\hat{\rho}^{\dagger}_{ss'\mathbf{kq}} =& \frac{i}{\hbar}\Delta\xi^{ss'}_{\mathbf{kq}}\hat{\rho}^{\dagger}_{ss'\mathbf{kq}} + \frac{iV_q}{\hbar S}\langle s'\mathbf{k}|s\mathbf{k}+\mathbf{q}\rangle\sum_{ll'\mathbf{k}}\Delta n^{ll'}_{\mathbf{k'q}}\langle l\mathbf{k'}+\mathbf{q}|l'\mathbf{k'}\rangle\hat{\rho}^{l\dagger}_{ll'\mathbf{k'q}} \\
&- ie\sqrt{\frac{\pi}{\nu}}\sum_{p_z}\sqrt{\frac{2}{\hbar\omega_{\mathbf{q}p_z}}}\Delta n^{ss'}_{\mathbf{kq}}\{\langle s'\mathbf{k}|(J^y_{\mathbf{kq}})^{\dagger}|s\mathbf{k}+\mathbf{q}\rangle(\hat{a}_{1,(-\mathbf{q},-p_z)} - \hat{a}^{\dagger}_{1,(\mathbf{q},p_z)}) \\
&+ \frac{p_z}{\sqrt{q^2+p_z^2}}\langle s'\mathbf{k}|(J^x_{\mathbf{kq}})^{\dagger}|s\mathbf{k}+\mathbf{q}\rangle(\hat{a}_{2,(-\mathbf{q},-p_z)} + \hat{a}^{\dagger}_{2,(\mathbf{q},p_z)})\}
\end{aligned} \tag{20}$$

For photons, we can set

$$\begin{aligned}
&\xi_1(\mathbf{q},p_z) = (-\sin\theta_{\mathbf{q}}, \cos\theta_{\mathbf{q}}, 0), \\
&\xi_2(\mathbf{q},p_z) = (-\cos\phi_{\mathbf{q}}\cos\theta_{\mathbf{q}}, -\cos\phi_{\mathbf{q}}\sin\theta_{\mathbf{q}}, \sin\phi_{\mathbf{q}})
\end{aligned} \tag{21}$$

Because the single electron Hamiltonian has rotation symmetry, once we have chosen an in-plane wave vector $\mathbf{q}$, we can use its direction as the x axis, now $\theta_{(\mathbf{q},p_z)} = 0$, $\theta_{(-\mathbf{q},-p_z)} = \pi$, and

$$\begin{aligned}
&\xi_2(\mathbf{q},p_z) = \xi_2(-\mathbf{q},-p_z) = -\cos\phi_{(\mathbf{q},p_z)}\mathbf{i} + \sin\phi_{(\mathbf{q},p_z)}\mathbf{k} \\
&\xi_1(\mathbf{q},p_z) = -\xi_1(-\mathbf{q},-p_z) = \mathbf{j}
\end{aligned} \tag{22}$$

then the time evolution for photons read

$$\begin{aligned}
\partial_t\hat{a}_{1,(-\mathbf{q},-p_z)} =& -i\omega_{qp_z}\hat{a}_{1(-\mathbf{q},-p_z)} + ie\sqrt{\frac{\pi}{\nu}}\sum_{ss'\mathbf{k}}\sqrt{\frac{2}{\hbar\omega_{qp_z}}}\langle s\mathbf{k}+\mathbf{q}|J^y_{\mathbf{kq}}|s'\mathbf{k}\rangle\hat{\rho}^{\dagger}_{ss'\mathbf{kq}} \\
&- \sqrt{\frac{1}{\hbar\omega_{qp_z}}}\frac{2\pi i\hbar\Pi^d}{L_z}\sum_{p'_z}\sqrt{\frac{1}{\hbar\omega_{qp'_z}}}(\hat{a}_{1,-\mathbf{q},-p'_z} - \hat{a}^{\dagger}_{1\mathbf{q}p'_z}) \\
\partial_t\hat{a}_{2,(-\mathbf{q},-p_z)} =& -i\omega_{\mathbf{q}p_z}\hat{a}_{2,(-\mathbf{q},-p_z)} + ie\sqrt{\frac{\pi}{\nu}}\sum_{ss'\mathbf{k}}\sqrt{\frac{2}{\hbar\omega_{\mathbf{q}p_z}}}\frac{q_z}{\sqrt{q^2+q_z^2}}\langle s\mathbf{k}+\mathbf{q}|J^x_{\mathbf{kq}}|s'\mathbf{k}\rangle\hat{\rho}^{\dagger}_{ss'\mathbf{kq}} \\
&- \sqrt{\frac{1}{\hbar\omega_{qp_z}}}\frac{p_z}{\sqrt{q^2+p_z^2}}\frac{2\pi i\hbar\Pi^d}{L_z}\sum_{p'_z}\sqrt{\frac{1}{\hbar\omega_{qp'_z}}}\frac{p'_z}{\sqrt{q^2+p'^2_z}}(\hat{a}_{2,-\mathbf{q},-p'_z} + \hat{a}^{\dagger}_{2\mathbf{q}p'_z}) \\
&- \sqrt{\frac{1}{\hbar\omega_{qp_z}}}\frac{q}{\sqrt{q^2+p_z^2}}\frac{2\pi i\hbar\Pi^d}{L_z}\sum_{p'_z}\sqrt{\frac{1}{\hbar\omega_{qp'_z}}}\frac{q}{\sqrt{q^2+p'^2_z}}(\hat{a}_{2,-\mathbf{q},-p'_z} + \hat{a}^{\dagger}_{2\mathbf{q}p'_z})
\end{aligned} \tag{23}$$

and

$$\partial_t \hat{a}^\dagger_{1,(\mathbf{q},p_z)} = i\omega_{qp_z}\hat{a}^\dagger_{1(\mathbf{q},p_z)} + ie\sqrt{\frac{\pi}{\nu}}\sum_{ss'\mathbf{k}}\sqrt{\frac{2}{\hbar\omega_{qp_z}}}\langle s\mathbf{k}+\mathbf{q}|J^y_{\mathbf{kq}}|s'\mathbf{k}\rangle\hat{\rho}^\dagger_{ss'\mathbf{kq}}$$
$$-\sqrt{\frac{1}{\hbar\omega_{qp_z}}}\frac{2\pi i\hbar\Pi^d}{L_z}\sum_{p'_z}\sqrt{\frac{1}{\hbar\omega_{qp'_z}}}(\hat{a}_{1,-\mathbf{q},-p'_z} - \hat{a}^\dagger_{1\mathbf{q}p'_z})$$
$$\partial_t \hat{a}^\dagger_{2,(\mathbf{q},p_z)} = i\omega_{\mathbf{q}p_z}\hat{a}^\dagger_{2,(\mathbf{q},p_z)} - ie\sqrt{\frac{\pi}{\nu}}\sum_{ss'\mathbf{k}}\sqrt{\frac{2}{\hbar\omega_{\mathbf{q}p_z}}}\frac{q_z}{\sqrt{q^2+q_z^2}}\langle s\mathbf{k}+\mathbf{q}|J^x_{\mathbf{kq}}|s'\mathbf{k}\rangle\hat{\rho}^\dagger_{ss'\mathbf{kq}}$$
$$+\sqrt{\frac{1}{\hbar\omega_{qp_z}}}\frac{p_z}{\sqrt{q^2+p_z^2}}\frac{2\pi i\hbar\Pi^d}{L_z}\sum_{p'_z}\sqrt{\frac{1}{\hbar\omega_{qp'_z}}}\frac{p'_z}{\sqrt{q^2+p'^2_z}}(\hat{a}_{2,-\mathbf{q},-p'_z} + \hat{a}^\dagger_{2\mathbf{q}p'_z})$$
$$+\sqrt{\frac{1}{\hbar\omega_{qp_z}}}\frac{q}{\sqrt{q^2+p_z^2}}\frac{2\pi i\hbar\Pi^d}{L_z}\sum_{p'_z}\sqrt{\frac{1}{\hbar\omega_{qp'_z}}}\frac{q}{\sqrt{q^2+p'^2_z}}(\hat{a}_{2,-\mathbf{q},-p'_z} + \hat{a}^\dagger_{2\mathbf{q}p'_z}) \tag{24}$$

III. COLLECTIVE MODES

The collective modes can be expressed as the summation of electron density oscillations, photon creations and annihilations,

$$\hat{Q}^\dagger_{\mathbf{q}} = \sum_{s,s'}\sum_{\mathbf{k}}\Phi_{ss'\mathbf{kq}}\hat{\rho}^\dagger_{ss'\mathbf{kq}} + \sum_{\lambda=1,2}\sum_{p_z}\Phi^a_{\lambda,p_z}\hat{a}_{\lambda,(-\mathbf{q},-p_z)} + \Phi^c_{\lambda,p_z}\hat{a}^\dagger_{\lambda,(\mathbf{q},p_z)} \tag{25}$$

Like Nambu spinor, let's define $\hat{\Phi}^\dagger_{\mathbf{q}} = [\cdots\hat{\rho}^\dagger_{\mathbf{k}_i\mathbf{q}s_is'_i,\Delta n_i=0}\cdots\hat{\rho}^\dagger_{\mathbf{k}_i\mathbf{q}s_is'_i,\Delta n_i\neq 0}\cdots\hat{a}_{\lambda(-\mathbf{q},-p_{zi})}\cdots\hat{a}^\dagger_{\lambda(\mathbf{q},p_{zi})}\cdot\cdot]^T$, where $\lambda=1,2$, $s_is'_i = ++,+-,-+,--$. And we can rewrite the collective mode with

$$\hat{Q}^\dagger_{\mathbf{q}} = \Phi_{\mathbf{q}}\hat{\Phi}^\dagger_{\mathbf{q}} \tag{26}$$

$\Phi_{\mathbf{q}} = [\cdots\Phi_{s_is'_i\mathbf{k}_i\mathbf{q},\Delta n_i=0}\cdots\Phi_{s_is'_i\mathbf{k}_i\mathbf{q},\Delta n_i\neq 0}\cdots\Phi^a_{\lambda,p_z}\cdots\Phi^c_{\lambda,p_z}\cdot\cdot]$ is then the wave function. For convenience, we put density oscillation operators that satisfy $\Delta n_i = \Delta n^{s_is'_i}_{\mathbf{k}_i\mathbf{q}} = 0$ in the head, which followed by $\hat{\rho}^\dagger_{\mathbf{k}_i\mathbf{q}s_is'_i}$ with $\Delta n_i \neq 0$.

Then we can compare the coefficients of different operators in the time evolution $\partial_t\hat{Q}^\dagger_{\mathbf{q}} = i\Omega_{\mathbf{q}}\hat{Q}^\dagger_{\mathbf{q}}$, and get

$$\hbar\Omega_{\mathbf{q}}\Phi_{ss'\mathbf{kq}} = \sum_{ll'\mathbf{k}'}[\delta_{ls}\delta_{l's'}\delta_{\mathbf{kk}'}(\Delta\xi^{ll'}_{\mathbf{k}'\mathbf{q}} - i\delta)\Phi_{ll'\mathbf{k}'\mathbf{q}} + \frac{V_q}{S}\langle s\mathbf{k}+\mathbf{q}|s'\mathbf{k}\rangle\langle l'\mathbf{k}'|l\mathbf{k}'+\mathbf{q}\rangle\Delta n^{ll'}_{\mathbf{k}'\mathbf{q}}\Phi_{ll'\mathbf{k}'\mathbf{q}}]$$
$$+\hbar e\langle s\mathbf{k}+\mathbf{q}|J^y_{\mathbf{kq}}|s'\mathbf{k}\rangle\sum_{p_z}\sqrt{\frac{2\pi}{\hbar\omega_{\mathbf{q}p_z}\nu}}(\Phi^a_{1p_z} + \Phi^c_{1p_z}) + \hbar e\langle s\mathbf{k}+\mathbf{q}|J^x_{\mathbf{kq}}|s'\mathbf{k}\rangle\sum_{p_z}\sqrt{\frac{2\pi}{\hbar\omega_{\mathbf{q}p_z}\nu}}\frac{p_z}{\sqrt{q^2+p_z^2}}(\Phi^a_{2p_z} - \Phi^c_{2p_z}) \tag{27}$$

For photons, we have

$$\hbar\Omega_{\mathbf{q}}\Phi^a_{1,p_z} = -\sum_{p'_z}\delta_{p_z,p'_z}(\hbar\omega_{\mathbf{q}p'_z} + i\delta)\Phi^a_{1,p'_z} - \frac{2\pi\hbar e^2\Pi_d}{L_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_z}}}\sum_{p'_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p'_z}}}(\Phi^a_{1,p'_z} + \Phi^c_{1,p'_z})$$
$$-e\sum_{ss'\mathbf{k}}\sqrt{\frac{2\pi}{\hbar\omega_{\mathbf{q}p_z}\nu}}\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|\hbar(J^y_{\mathbf{kq}})^\dagger|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{kq}}$$
$$\hbar\Omega_{\mathbf{q}}\Phi^c_{1,p_z} = \sum_{p'_z}\delta_{p_z,p'_z}(\hbar\omega_{\mathbf{q}p'_z} + i\delta)\Phi^c_{1,p'_z} + \frac{2\pi\hbar e^2\Pi_d}{L_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_z}}}\sum_{p'_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p'_z}}}(\Phi^a_{1,p'_z} + \Phi^c_{1,p'_z})$$
$$+e\sum_{ss'\mathbf{k}}\sqrt{\frac{2\pi}{\hbar\omega_{\mathbf{q}p_z}\nu}}\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|\hbar(J^y_{\mathbf{kq}})^\dagger|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{kq}} \tag{28}$$

and

$$
\begin{aligned}
\hbar\Omega_{\mathbf{q}}\Phi^a_{2,p_z} =& -\sum_{p'_z}\delta_{p_z,p'_z}(\hbar\omega_{\mathbf{q}p'_z}+i\delta)\Phi^a_{2,p'_z} - \frac{2\pi\hbar e^2\Pi_d}{L_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_z}}}\frac{p_z}{\sqrt{q^2+p_z^2}}\sum_{p'_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p'_z}}}\frac{p'_z}{\sqrt{q^2+p'^2_z}}(\Phi^a_{2,p'_z}-\Phi^c_{2,p'_z})\\
&-\frac{2\pi\hbar e^2\Pi_d}{L_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_z}}}\frac{q}{\sqrt{q^2+p_z^2}}\sum_{p'_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p'_z}}}\frac{q}{\sqrt{q^2+p'^2_z}}(\Phi^a_{2,p'_z}-\Phi^c_{2,p'_z})\\
&-\frac{ep_z}{\sqrt{q^2+p_z^2}}\sqrt{\frac{2\pi}{\hbar\omega_{\mathbf{q}p_z}\nu}}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|\hbar(J^x_{\mathbf{kq}})^\dagger|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{kq}}\\
\hbar\Omega_{\mathbf{q}}\Phi^c_{2,p_z} =& \sum_{p'_z}\delta_{p_z,p'_z}(\hbar\omega_{\mathbf{q}p'_z}+i\delta)\Phi^a_{2,p'_z} - \frac{2\pi\hbar e^2\Pi_d}{L_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_z}}}\frac{p_z}{\sqrt{q^2+p_z^2}}\sum_{p'_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p'_z}}}\frac{p'_z}{\sqrt{q^2+p'^2_z}}(\Phi^a_{2,p'_z}-\Phi^c_{2,p'_z})\\
&-\frac{2\pi\hbar e^2\Pi_d}{L_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_z}}}\frac{q}{\sqrt{q^2+p_z^2}}\sum_{p'_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p'_z}}}\frac{q}{\sqrt{q^2+p'^2_z}}(\Phi^a_{2,p'_z}-\Phi^c_{2,p'_z})\\
&-\frac{ep_z}{\sqrt{q^2+p_z^2}}\sqrt{\frac{2\pi}{\hbar\omega_{\mathbf{q}p_z}\nu}}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|\hbar(J^x_{\mathbf{kq}})^\dagger|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{kq}} \qquad (29)
\end{aligned}
$$

IV. PROPERTIES OF THE EIGEN EQUATIONS

Before solving these coupled linear equations Eq. (27-29), we will firstly prove some general properties for them. These equations are equivalent to a Schrödinger like equation $\bar{H}_{\mathbf{q}}\Phi_{\mathbf{q}} = (\hbar\Omega_{\mathbf{q}}+i\delta)\Phi_{\mathbf{q}}$, where

$$
\bar{H}_{\mathbf{q}} = \begin{pmatrix}
\bar{H}^{\rho\rho}_{\mathbf{q}} & \bar{H}^{\rho a_1}_{\mathbf{q}} & \bar{H}^{\rho a_2}_{\mathbf{q}} & \bar{H}^{\rho c_1}_{\mathbf{q}} & \bar{H}^{\rho c_2}_{\mathbf{q}}\\
\bar{H}^{a_1\rho}_{\mathbf{q}} & \bar{H}^{a_1 a_1}_{\mathbf{q}} & 0 & \bar{H}^{a_1 c_1}_{\mathbf{q}} & 0\\
\bar{H}^{a_2\rho}_{\mathbf{q}} & 0 & \bar{H}^{a_2 a_2}_{\mathbf{q}} & 0 & \bar{H}^{a_2 c_2}_{\mathbf{q}}\\
\bar{H}^{c_1\rho}_{\mathbf{q}} & \bar{H}^{c_1 a_1}_{\mathbf{q}} & 0 & \bar{H}^{c_1 c_1}_{\mathbf{q}} & 0\\
\bar{H}^{c_2\rho}_{\mathbf{q}} & 0 & \bar{H}^{c_2 a_2}_{\mathbf{q}} & 0 & \bar{H}^{c_2 c_2}_{\mathbf{q}}
\end{pmatrix} \qquad (30)
$$

where

$$
\begin{aligned}
&[\bar{H}^{\rho\rho}_{\mathbf{q}}]_{ij} = \delta_{s_i s_j}\delta_{s'_i s'_j}\delta_{\mathbf{k}_i\mathbf{k}'_j}(\Delta\xi^{s_j s'_j}_{\mathbf{k}_j\mathbf{q}} - i\delta) + \frac{V_q}{S}\langle s_i\mathbf{k}_i+\mathbf{q}|s'_i\mathbf{k}\rangle\langle s'_j\mathbf{k}_j|s_j\mathbf{k}_j+\mathbf{q}\rangle\Delta n^{s_j s'_j}_{\mathbf{k}_j\mathbf{q}}\\
&[\bar{H}^{\rho a_1}_{\mathbf{q}}]_{ij} = [\bar{H}^{\rho c_1}_{\mathbf{q}}]_{ij} = \hbar e\langle s_i\mathbf{k}_i+\mathbf{q}|J^y_{\mathbf{kq}}|s'_i\mathbf{k}_i\rangle\sqrt{\frac{2\pi}{\hbar\omega_{\mathbf{q}p_{zj}}\nu}}\\
&[\bar{H}^{\rho a_2}_{\mathbf{q}}]_{ij} = -[\bar{H}^{\rho c_2}_{\mathbf{q}}]_{ij} = \hbar e\langle s_i\mathbf{k}_i+\mathbf{q}|J^x_{\mathbf{kq}}|s'_i\mathbf{k}_i\rangle\sqrt{\frac{2\pi}{\hbar\omega_{\mathbf{q}p_{zj}}\nu}}\frac{p_{zj}}{\sqrt{q^2+p_{zj}^2}}
\end{aligned} \qquad (31)
$$

and

$$
\begin{aligned}
&[\bar{H}^{a_1\rho}_{\mathbf{q}}]_{ij} = -[\bar{H}^{c_1\rho}_{\mathbf{q}}]_{ij} = -e\sqrt{\frac{2\pi}{\hbar\omega_{\mathbf{q}p_{z_i}}\nu}}\Delta n^{s_j s'_j}_{\mathbf{k}_j\mathbf{q}}\langle s'_j\mathbf{k}_j|\hbar(J^y_{\mathbf{kq}})^\dagger|s_j\mathbf{k}_j+\mathbf{q}\rangle\\
&[\bar{H}^{a_1 a_1}_{\mathbf{q}}]_{ij} = -[\bar{H}^{c_1 c_1}_{\mathbf{q}}]_{ij} = -\delta_{p_{z_i}p_{z_j}}(\hbar\omega_{\mathbf{q}p_{z_j}}+i\delta) - \frac{2\pi\hbar e^2\Pi_d}{L_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_{z_i}}\omega_{\mathbf{q}p_{z_j}}}}\\
&[\bar{H}^{a_1 c_1}_{\mathbf{q}}]_{ij} = -[\bar{H}^{c_1 a_1}_{\mathbf{q}}]_{ij} = -\frac{2\pi\hbar e^2\Pi_d}{L_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_{z_i}}\omega_{\mathbf{q}p_{z_j}}}}
\end{aligned} \qquad (32)
$$

$$[\bar{H}_{\mathbf{q}}^{a_2\rho}]_{ij} = [\bar{H}_{\mathbf{q}}^{c_2\rho}]_{ij} = -\frac{ep_{z_i}}{\sqrt{q^2+p_{z_i}^2}}\sqrt{\frac{2\pi}{\hbar\omega_{\mathbf{q}p_{z_i}}\nu}}\Delta n_{\mathbf{k}_j\mathbf{q}}^{s_js_j'}\langle s_j'\mathbf{k}_j|\hbar(J_{\mathbf{kq}}^x)^\dagger|s_j\mathbf{k}_j+\mathbf{q}\rangle$$
$$[\bar{H}_{\mathbf{q}}^{a_2a_2}]_{ij} = -[\bar{H}_{\mathbf{q}}^{c_2c_2}]_{ij} = -\delta_{p_{z_i}p_{z_j}}(\hbar\omega_{\mathbf{q}p_{z_j}}+i\delta) - \frac{2\pi\hbar e^2\Pi_d}{L_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_{z_i}}\omega_{\mathbf{q}p_{z_j}}}}\frac{p_{z_i}}{\sqrt{q^2+p_{z_i}^2}}\frac{ep_{z_j}}{\sqrt{q^2+p_{z_j}^2}}$$
$$[\bar{H}_{\mathbf{q}}^{a_1c_1}]_{ij} = -[\bar{H}_{\mathbf{q}}^{c_1a_1}]_{ij} = \frac{2\pi\hbar e^2\Pi_d}{L_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_{z_i}}\omega_{\mathbf{q}p_{z_j}}}}\frac{p_{z_i}}{\sqrt{q^2+p_{z_i}^2}}\frac{ep_{z_j}}{\sqrt{q^2+p_{z_j}^2}} \tag{33}$$

we can prove that

$$-I_{ph}\bar{H}_{\mathbf{q}}^{a_1\rho} = \bar{H}_{\mathbf{q}}^{\rho a_1\dagger}J_e, \quad -I_{ph}\bar{H}_{\mathbf{q}}^{a_2\rho} = \bar{H}_{\mathbf{q}}^{\rho a_2\dagger}J_e,$$
$$I_{ph}\bar{H}_{\mathbf{q}}^{c_1\rho} = \bar{H}_{\mathbf{q}}^{\rho c_1\dagger}J_e, \quad I_{ph}\bar{H}_{\mathbf{q}}^{c_2\rho} = \bar{H}_{\mathbf{q}}^{\rho c_2\dagger}J_e, \tag{34}$$

where I_{ph} is the unit matrix with the same size of $\bar{H}_{\mathbf{q}}^{a_ia_i}$ or $\bar{H}_{\mathbf{q}}^{c_ic_i}$ $(i=1,2)$, and $J_e = \mathrm{diag}(...\Delta n_{\mathbf{k}_i\mathbf{q}}^{s_is_i'}...)$. Note that $\Delta n_{\mathbf{k}_j\mathbf{q}}^{s_js_j'}[\bar{H}_{\mathbf{q}}^{\rho\rho}]_{ji} = \Delta n_{\mathbf{k}_i\mathbf{q}}^{s_is_i'}[\bar{H}_{\mathbf{q}}^{\rho\rho}]_{ij}$ which means $\bar{H}_{\mathbf{q}}^{\rho\rho\dagger}J_e = J_e\bar{H}_{\mathbf{q}}^{\rho\rho}$, we find

$$\bar{H}_{\mathbf{q}}^\dagger J = J\bar{H}_{\mathbf{q}} \tag{35}$$

where

$$J = \begin{pmatrix} J_e & 0 & 0 & 0 & 0 \\ 0 & -I_{ph} & 0 & 0 & 0 \\ 0 & 0 & -I_{ph} & 0 & 0 \\ 0 & 0 & 0 & I_{ph} & 0 \\ 0 & 0 & 0 & 0 & I_{ph} \end{pmatrix} \tag{36}$$

Note that $\bar{H}_{\mathbf{q}}$ isn't hermitian, with the help of Eq. (35), we can prove $\hbar\Omega_{n\mathbf{q}}\langle\Phi_{m\mathbf{q}}|J|\Phi_{n\mathbf{q}}\rangle = \langle\Phi_{m\mathbf{q}}|J\bar{H}_{\mathbf{q}}|\Phi_{n\mathbf{q}}\rangle = \langle\Phi_{m\mathbf{q}}|\bar{H}_{\mathbf{q}}^\dagger J|\Phi_{n\mathbf{q}}\rangle = \hbar\Omega_{m\mathbf{q}}^*\langle\Phi_{m\mathbf{q}}|J|\Phi_{n\mathbf{q}}\rangle$. Immediately, one find

$$\langle\Phi_{m\mathbf{q}}|J|\Phi_{n\mathbf{q}}\rangle = 0, \quad m \neq n \tag{37}$$

when the spectrum is none degenerate. For convenience, we can put states satisfying $\Delta n_{\mathbf{k}_i\mathbf{q}}^{s_is_i'} = 0$ in the front of $\Phi_{\mathbf{q}}$ (suppose their total number is $2M$). It is easily to check that the first $2M$ eigenstates are just the first $2M$ basis states. And we have

$$\langle\Phi_{m\mathbf{q}}|J|\Phi_{m\mathbf{q}}\rangle = 0, \quad |m| \leqslant M \tag{38}$$

Then we can define two matrix, R_{el} with the same size of $\bar{H}_{\mathbf{q}}^{\rho\rho}$ exchanging $\mathbf{k}_i, s_i, s_i'$ with $\mathbf{k}_i+\mathbf{q}, s_i', s_i$ in the wave function $\Phi_{\mathbf{q}}^\rho$, and

$$R = \begin{pmatrix} R_{el} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{ph} & 0 \\ 0 & 0 & 0 & 0 & I_{ph} \\ 0 & I_{ph} & 0 & 0 & 0 \\ 0 & 0 & I_{ph} & 0 & 0 \end{pmatrix} \tag{39}$$

One can easily check $R^{-1} = R^\dagger = R$ and

$$R^\dagger\bar{H}_{\mathbf{q}}R = -\bar{H}_{-\mathbf{q}}^*, \tag{40}$$

this relation means $\bar{H}_{\mathbf{q}}$ has the particle-hole symmetry that if $\Phi_{n\mathbf{q}}$ is a wave function with $\hbar\Omega_{n\mathbf{q}}$, then $R\Phi_{n\mathbf{q}}^*$ is the wave function of $\bar{H}_{-\mathbf{q}}$ with $-\hbar\Omega_{n\mathbf{q}}^*$. So, we have $R\Phi_{n\mathbf{q}}^* = c\Phi_{-n,-\mathbf{q}}$, where $-n$ means the real part of quasi particle energy is negative, and c is a constant satisfying $|c|^2 = 1$, and can be chosen to 1. Now

$$\hat{Q}_{-n,-\mathbf{q}} = \hat{Q}_{n,\mathbf{q}}^\dagger \tag{41}$$

Because $R^\dagger J R = -J$, we have $\langle \Phi_{-n,-\mathbf{q}}|J|\Phi_{-n,-\mathbf{q}}\rangle = -\langle \Phi_{n\mathbf{q}}|J|\Phi_{n\mathbf{q}}\rangle^*$, and one can prove the requirement of $[\hat{Q}_{n\mathbf{q}}, \hat{Q}^\dagger_{m\mathbf{q}}] = \delta_{nm}$ (quasi particles should be bosons) equals to the orthogonalization (Eq. (37, 38)) requirements together with the following normalization conditions

$$\begin{aligned} \langle \Phi_{n\mathbf{q}}|J|\Phi_{n\mathbf{q}}\rangle &= 1, \quad |n| > M, \Omega_n > 0 \\ \langle \Phi_{n\mathbf{q}}|J|\Phi_{n\mathbf{q}}\rangle &= -1, \quad |n| > M, \Omega_n < 0 \end{aligned} \tag{42}$$

If we find all the eigen values $\hbar\Omega_{\pm n\mathbf{q}}$ and wave functions $\Phi_{\pm n\mathbf{q}}$ ($n = 1, 2, ...M + N$, total dimension of $\bar{H}_\mathbf{q}$ is 2(M+N)). One have the following matrix

$$U = \begin{pmatrix} \Phi^T_{-1\mathbf{q}} \\ \Phi^T_{-2\mathbf{q}} \\ . \\ . \\ . \\ \Phi^T_{-M\mathbf{q}} \\ \Phi^T_{1\mathbf{q}} \\ \Phi^T_{2\mathbf{q}} \\ . \\ . \\ . \\ \Phi^T_{M\mathbf{q}} \\ \Phi^T_{-(M+1)\mathbf{q}} \\ \Phi^T_{-(M+2)\mathbf{q}} \\ . \\ . \\ . \\ \Phi^T_{-(M+N)\mathbf{q}} \\ \Phi^T_{M+1\mathbf{q}} \\ \Phi^T_{M+2\mathbf{q}} \\ . \\ . \\ . \\ \Phi^T_{M+N\mathbf{q}} \end{pmatrix} = \begin{pmatrix} I_{2M\times 2M} & 0 \\ U_z & U_{nz} \end{pmatrix} \tag{43}$$

(the superscript ^{T} means transpose), we have

$$UJU^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -I_{N\times N} & 0 \\ 0 & 0 & I_{N\times N} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma_3 \otimes I_N \end{pmatrix}. \tag{44}$$

If choosing an even N, one can prove $\det|U_{nz}J_{nz}U^\dagger_{nz}| = \det|J_{nz}| \times |\det|U_{nz}||^2 = 1$ (J_{nz} is the none zero sub-matrix of J_e). And because $\det|J_{nz}| > 0$, we have $\det|U_{nz}| \neq 0$ and get

$$U^{-1}_{nz} = -J_{nz}U^\dagger_{nz}\sigma_3 \otimes I_N. \tag{45}$$

For the whole matrix U, we have

$$U^{-1} = \begin{pmatrix} I_{2M\times 2M} & 0 \\ -J_{nz}U^\dagger_{nz}\sigma_3 \otimes I_N U_z & -J_{nz}U^\dagger_{nz}\sigma_3 \otimes I_N \end{pmatrix}. \tag{46}$$

Finally, we can derive the following inverse transformation

$$\hat{\Phi}^{\dagger}_{\mathbf{q}} = U^{-1} \begin{pmatrix} \hat{Q}^{\dagger}_{1,\mathbf{q}} \\ \cdot \\ \cdot \\ \cdot \\ \hat{Q}^{\dagger}_{M,\mathbf{q}} \\ \hat{Q}_{1,-\mathbf{q}} \\ \cdot \\ \cdot \\ \cdot \\ \hat{Q}_{M,-\mathbf{q}} \\ \hat{Q}^{\dagger}_{M+1,\mathbf{q}} \\ \cdot \\ \cdot \\ \cdot \\ \hat{Q}^{\dagger}_{M+N,\mathbf{q}} \\ \hat{Q}_{M+1,-\mathbf{q}} \\ \cdot \\ \cdot \\ \cdot \\ \hat{Q}_{M+N,-\mathbf{q}} \end{pmatrix}. \tag{47}$$

V. ENERGIES AND WAVE FUNCTIONS OF QUASI PARTICLES

In this section, let's solve the coupled linear equations Eq. (27-29). To do so, we can firstly define

$$\begin{aligned} S_{1\mathbf{q}} &= \frac{1}{L_z}\sum_{p_z}\sqrt{\frac{1}{\omega_{\mathbf{q}p_z}}}(\Phi^a_{1,p_z}+\Phi^c_{1,p_z}) \\ S_{2\mathbf{q}} &= \frac{1}{L_z}\sum_{p_z}\frac{p_z}{\sqrt{q^2+p_z^2}}\sqrt{\frac{1}{\omega_{\mathbf{q}p_z}}}(\Phi^a_{2,p_z}-\Phi^c_{2,p_z}) \\ S_{3\mathbf{q}} &= \frac{1}{L_z}\sum_{p_z}\frac{q}{\sqrt{q^2+p_z^2}}\sqrt{\frac{1}{\omega_{\mathbf{q}p_z}}}(\Phi^a_{2,p_z}-\Phi^c_{2,p_z}) \end{aligned} \tag{48}$$

One can easily check

$$\begin{aligned} S_{1\mathbf{q}} =& \frac{1}{L_z}\sum_{p_z}\frac{1}{\omega_{\mathbf{q}p_z}}\left[\frac{1}{\hbar\Omega_{\mathbf{q}}-\hbar\omega_{\mathbf{q}p_z}-i\delta}-\frac{1}{\hbar\Omega_{\mathbf{q}}+\hbar\omega_{\mathbf{q}p_z}+i\delta}\right]\{2\pi\hbar e^2\Pi_d S_{1\mathbf{q}} \\ &+ e\sqrt{\frac{2\pi}{\hbar\nu}}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{k}\mathbf{q}}\langle s'\mathbf{k}|\hbar(J^y_{\mathbf{k}\mathbf{q}})^{\dagger}|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{k}\mathbf{q}}\} \end{aligned} \tag{49}$$

note that

$$\frac{1}{L_z}\sum_{p_z}\frac{1}{\omega_{\mathbf{q}p_z}}\left[\frac{1}{\hbar\Omega_{\mathbf{q}}-\hbar\omega_{\mathbf{q}p_z}-i\delta}-\frac{1}{\hbar\Omega_{\mathbf{q}}+\hbar\omega_{\mathbf{q}p_z}+i\delta}\right] = -\frac{1}{c^2\pi\hbar}\int_{-\infty}^{\infty}\frac{1}{q_z^2+q^2-\Omega^2_{\mathbf{q}}/c^2}dp_z = -\frac{1}{\hbar c^2 q'} \tag{50}$$

so, we have

$$\left(1+\frac{q^d}{q'}\right)S_{1\mathbf{q}} = -\frac{e}{c^2q'}\sqrt{\frac{2\pi}{\hbar\nu}}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{k}\mathbf{q}}\langle s'\mathbf{k}|(J^y_{\mathbf{k}\mathbf{q}})^{\dagger}|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{k}\mathbf{q}} \tag{51}$$

where

$$q^d = 2\pi e^2\Pi_d/c^2, \quad q' = \sqrt{q^2-\Omega^2_{\mathbf{q}}/c^2} \tag{52}$$

Also, one can find $S_{3\mathbf{q}}=0$ and

$$
\begin{aligned}
S_{2\mathbf{q}} =&\frac{1}{L_z}\sum_{p_z}\frac{1}{\omega_{\mathbf{q}p_z}}\frac{p_z^2}{q^2+p_z^2}\left[\frac{1}{\hbar\Omega_{\mathbf{q}}-\hbar\omega_{\mathbf{q}p_z}-i\delta}-\frac{1}{\hbar\Omega_{\mathbf{q}}+\hbar\omega_{\mathbf{q}p_z}+i\delta}\right]\{2\pi\hbar e^2\Pi_d S_{2\mathbf{q}} \\
&+e\sqrt{\frac{2\pi}{\hbar\nu}}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|\hbar(J^x_{\mathbf{kq}})^\dagger|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{kq}}\} \\
=&-\frac{1}{\hbar c^2(q+q')}\{2\pi\hbar e^2\Pi_d S_{2\mathbf{q}}+e\sqrt{\frac{2\pi}{\hbar\nu}}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|\hbar(J^x_{\mathbf{kq}})^\dagger|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{kq}}\}
\end{aligned}
\tag{53}
$$

which means

$$
\left(1+\frac{q^d}{q+q'}\right)S_{2\mathbf{q}}=-\frac{e}{c^2(q+q')}\sqrt{\frac{2\pi}{\hbar\nu}}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|(J^x_{\mathbf{kq}})^\dagger|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{kq}}
\tag{54}
$$

Then we have

$$
\begin{aligned}
\Phi_{ss'\mathbf{kq}} =&\frac{1}{\hbar\Omega_{\mathbf{q}}-\Delta\xi^{ss'}_{\mathbf{kq}}+i\delta}\Bigg\{\frac{V_q}{S}\langle s\mathbf{k}+\mathbf{q}|s'\mathbf{k}\rangle\sum_{ll'\mathbf{k}'}\langle l'\mathbf{k}'|l\mathbf{k}'+\mathbf{q}\rangle\Delta n^{ll'}_{\mathbf{k}'\mathbf{q}}\Phi_{ll'\mathbf{k}'\mathbf{q}} \\
&-\frac{2\pi e^2}{c^2(q'+q^d)}\langle s\mathbf{k}+\mathbf{q}|J^y_{\mathbf{kq}}|s'\mathbf{k}\rangle\frac{1}{S}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{k}_1\mathbf{q}}\langle s'\mathbf{k}_1|(J^y_{\mathbf{k}_1\mathbf{q}})^*|s\mathbf{k}_1+\mathbf{q}\rangle\Phi_{ss'\mathbf{k}_1\mathbf{q}} \\
&-\frac{2\pi e^2}{c^2(q+q'+q^d)}\langle s\mathbf{k}+\mathbf{q}|J^x_{\mathbf{kq}}|s'\mathbf{k}\rangle\frac{1}{S}\sum_{ss'\mathbf{k}_1}\Delta n^{ss'}_{\mathbf{k}_1\mathbf{q}}\langle s'\mathbf{k}_1|(J^x_{\mathbf{k}_1\mathbf{q}})^*|s\mathbf{k}_1+\mathbf{q}\rangle\Phi_{ss'\mathbf{k}_1\mathbf{q}}\Bigg\}
\end{aligned}
\tag{55}
$$

Let's define

$$
\begin{aligned}
&V^p_{\mathbf{q}la}=-\frac{2\pi e^2}{c^2(q+q'+q^d)},\quad V_{\mathbf{q}la}=-\frac{2\pi e^2}{c^2(q+q')},\\
&V_{\mathbf{q}lb}=\frac{q^2}{\Omega^2_{\mathbf{q}}}V_q,\quad V_{\mathbf{q}l}=V_{\mathbf{q}lq}+V_{\mathbf{q}lb}=\frac{2\pi e^2}{\Omega^2_{\mathbf{q}}}q'\\
&V^p_{\mathbf{q}t}=-\frac{2\pi e^2}{c^2(q'+q^d)},\quad V_{\mathbf{q}t}=-\frac{2\pi e^2}{c^2q'}
\end{aligned}
\tag{56}
$$

and

$$
\begin{aligned}
L^\rho_{\mathbf{q}}&=\frac{V_q}{S}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{kq}},\\
L^x_{\mathbf{q}}&=\frac{V^p_{\mathbf{q}la}}{S}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|(J^x_{\mathbf{kq}})^\dagger|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{kq}},\\
L^y_{\mathbf{q}}&=\frac{V^p_{\mathbf{q}t}}{S}\sum_{ss'\mathbf{k}}\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|(J^y_{\mathbf{kq}})^\dagger|s\mathbf{k}+\mathbf{q}\rangle\Phi_{ss'\mathbf{kq}}
\end{aligned}
\tag{57}
$$

We have

$$
\Phi_{ss'\mathbf{kq}}=\frac{1}{\hbar\Omega_{\mathbf{q}}-\Delta\xi^{ss'}_{\mathbf{kq}}+i\delta}\{\langle s\mathbf{k}+\mathbf{q}|s'\mathbf{k}\rangle L^\rho_{\mathbf{q}}+\langle s\mathbf{k}+\mathbf{q}|J^y_{\mathbf{kq}}|s'\mathbf{k}\rangle L_y+\langle s\mathbf{k}+\mathbf{q}|J^x_{\mathbf{kq}}|s'\mathbf{k}\rangle L_x\}
\tag{58}
$$

which means

$$
\begin{aligned}
L^\rho_{\mathbf{q}}=&\frac{V_q}{S}\sum_{ss'\mathbf{k}}\frac{\Delta n^{ss'}_{\mathbf{kq}}\langle s'\mathbf{k}|s\mathbf{k}+\mathbf{q}\rangle}{\hbar\Omega_{\mathbf{q}}-\Delta\xi^{ss'}_{\mathbf{kq}}+i\delta}\{\langle s\mathbf{k}+\mathbf{q}|s'\mathbf{k}\rangle L^\rho_{\mathbf{q}}+\langle s\mathbf{k}+\mathbf{q}|J^y_{\mathbf{kq}}|s'\mathbf{k}\rangle L_y+\langle s\mathbf{k}+\mathbf{q}|J^x_{\mathbf{kq}}|s'\mathbf{k}\rangle L_x\}\\
=&\frac{V_q}{S}\sum_{i\omega_n,\mathbf{k}}\mathrm{Tr}[G(i\omega_n+i\Omega_{\mathbf{q}},\mathbf{k}+\mathbf{q})(L^\rho_{\mathbf{q}}+J^x_{\mathbf{kq}}L_x+J^y_{\mathbf{kq}}L_y)G(i\omega_n,\mathbf{k})]\\
=&V_q[\Pi_{\rho\rho}L^\rho_{\mathbf{q}}+\Pi_{\rho x}L_x+\Pi_{\rho y}L_y]
\end{aligned}
\tag{59}
$$

Similarly, we can prove

$$L^x_{\mathbf{q}} = \frac{V^p_{\mathbf{q}la}}{S}\sum_{i\omega_n,\mathbf{k}} \mathrm{Tr}[J^x_{\mathbf{kq}} G(i\omega_n + i\Omega_{\mathbf{q}}, \mathbf{k}+\mathbf{q})(L^\rho_{\mathbf{q}} + J^x_{\mathbf{kq}} L_x + J^y_{\mathbf{kq}} L_y) G(i\omega_n, \mathbf{k})]$$
$$= V^p_{\mathbf{q}la}[\Pi_{x\rho} L^\rho_{\mathbf{q}} + \Pi^p_{xx} L_x + \Pi_{xy} L_y] \tag{60}$$

and

$$L^y_{\mathbf{q}} = V^p_{\mathbf{q}t}[\Pi_{y\rho} L^\rho_{\mathbf{q}} + \Pi_{yx} L_x + \Pi^p_{yy} L_y] \tag{61}$$

Replace Π^p_{xx}, Π^p_{yy} with $\Pi_{xx} - \Pi^d, \Pi_{yy} - \Pi^d$, we have

$$\begin{aligned} L^\rho_{\mathbf{q}} &= V_q[\Pi_{\rho\rho} L^\rho_{\mathbf{q}} + \Pi_{\rho x} L_x + \Pi_{\rho y} L_y] \\ L^x_{\mathbf{q}} &= V_{\mathbf{q}la}[\Pi_{x\rho} L^\rho_{\mathbf{q}} + \Pi_{xx} L_x + \Pi_{xy} L_y] \\ L^y_{\mathbf{q}} &= V_{\mathbf{q}t}[\Pi_{y\rho} L^\rho_{\mathbf{q}} + \Pi_{yx} L_x + \Pi_{yy} L_y] \end{aligned} \tag{62}$$

Finally, we can find the quasi particle energy is determined by

$$\begin{aligned} &\begin{vmatrix} 1 - V_q\Pi_{\rho\rho} & -V_q\Pi_{\rho x} & -V_q\Pi_{\rho y} \\ -V_{qla}\Pi_{x\rho} & 1 - V_{qla}\Pi_{xx} & -V_{qla}\Pi_{xy} \\ -V_{qt}\Pi_{y\rho} & -V_{qt}\Pi_{yx} & 1 - V_{qt}\Pi_{yy} \end{vmatrix} = 0 \\ \Rightarrow &- V_{qt}\Pi_{y\rho} \begin{vmatrix} -V_q\Pi_{\rho x} & -V_q\Pi_{\rho y} \\ 1 - V_{qla}\Pi_{xx} & -V_{qla}\Pi_{xy} \end{vmatrix} + V_{qt}\Pi_{yx} \begin{vmatrix} 1 - V_q\Pi_{\rho\rho} & -V_q\Pi_{\rho y} \\ -V_{qla}\Pi_{x\rho} & -V_{qla}\Pi_{xy} \end{vmatrix} \\ &+ (1 - V_{qt}\Pi_{yy}) \begin{vmatrix} 1 - V_q\Pi_{\rho\rho} & -V_q\Pi_{\rho x} \\ -V_{qla}\Pi_{x\rho} & 1 - V_{qla}\Pi_{xx} \end{vmatrix} = 0 \\ \Rightarrow &- V_{qt}V_q\Pi_{y\rho}\Pi_{\rho y} - V_{qt}V_{qla}\Pi_{yx}\Pi_{xy} + (1 - V_{qt}\Pi_{yy})(1 - V_q\Pi_{\rho\rho} - V_{qla}\Pi_{xx}) = 0 \\ \Rightarrow &V_{ql}V_{qt}|\Pi_{xy}|^2 = (1 - V_{ql}\Pi_{xx})(1 - V_{qt}\Pi_{yy}) \end{aligned} \tag{63}$$

Once getting the excitation energy, we have

$$\begin{aligned} L^x_{\mathbf{q}} &= \frac{\begin{vmatrix} 1 - V_q\Pi_{\rho\rho} & V_q\Pi_{\rho y} \\ -V_{qla}\Pi_{x\rho} & V_{qla}\Pi_{xy} \end{vmatrix}}{\begin{vmatrix} V_q\Pi_{\rho x} & V_q\Pi_{\rho y} \\ V_{qla}\Pi_{xx} - 1 & V_{qla}\Pi_{xy} \end{vmatrix}} L^\rho_{\mathbf{q}} = \frac{V_{qla}\Pi_{xy}}{V_q\Pi_{\rho y}} L^\rho_{\mathbf{q}} = \frac{q' - q}{\Omega_{\mathbf{q}}} L^\rho_{\mathbf{q}} \\ L^y_{\mathbf{q}} &= -\frac{\begin{vmatrix} 1 - V_q\Pi_{\rho\rho} & V_q\Pi_{\rho x} \\ -V_{qla}\Pi_{x\rho} & V_{qla}\Pi_{xx} - 1 \end{vmatrix}}{\begin{vmatrix} V_q\Pi_{\rho x} & V_q\Pi_{\rho y} \\ V_{qla}\Pi_{xx} - 1 & V_{qla}\Pi_{xy} \end{vmatrix}} L^\rho_{\mathbf{q}} = \frac{1 - V_{ql}\Pi_{xx}}{V_q\Pi_{\rho y}} L^\rho_{\mathbf{q}}, \end{aligned} \tag{64}$$

which means, for a pure longitudinal mode, $L^y_{\mathbf{q}} = 0$ because $1 - V_{ql}\Pi_{xx} = 0$. Now,

$$\Phi_{ss'\mathbf{kq}} = \frac{L^\rho_{\mathbf{q}}}{\hbar\Omega_{\mathbf{q}} - \Delta\xi^{ss'}_{\mathbf{kq}} + i\delta}\{\langle s\mathbf{k}+\mathbf{q}|s'\mathbf{k}\rangle + \langle s\mathbf{k}+\mathbf{q}|J^y_{\mathbf{kq}}|s'\mathbf{k}\rangle \frac{1 - V_{ql}\Pi_{xx}}{V_q\Pi_{\rho y}} + \langle s\mathbf{k}+\mathbf{q}|J^x_{\mathbf{kq}}|s'\mathbf{k}\rangle \frac{q' - q}{\Omega_{\mathbf{q}}}\} \tag{65}$$

and

$$\Phi^a_{1,p_z}, \Phi^c_{1,p_z} = \mp \frac{eL^y_{\mathbf{q}}}{\Omega_{\mathbf{q}} \pm \omega_{\mathbf{q}p_z} + i\delta/\hbar}\sqrt{\frac{2\pi S}{\hbar\omega_{\mathbf{q}q_z} L_z}}\frac{1}{V_{qt}} \tag{66}$$

$$\Phi^a_{2,p_z}, \Phi^c_{2,p_z} = -\frac{\omega_{\mathbf{q}} L^\rho_q}{2\pi e(\omega_{\mathbf{q}} \pm \omega_{\mathbf{q}p_z} + i\delta/\hbar)}\frac{p_z}{\sqrt{q^2 + p_z^2}}\sqrt{\frac{2\pi S}{\hbar\omega_{\mathbf{q}p_z} L_z}} \tag{67}$$

Note that in the wave functions, we can use either $L^x_{\mathbf{q}}$ or $L^\rho_{\mathbf{q}}$ which average determined by $[\hat{Q}_{\mathbf{q}}, \hat{Q}^\dagger_{\mathbf{q}'}] = \delta_{\mathbf{qq}'}$.

VI. INTERACTION WITH EMITTERS

In this section, we will add an emitter in our system, and calculate the interactions between emitters with SPPs. Suppose we have a 3D or 2D emitter (e.g. hydrogen like atoms) localized near the 2D electron gas, the total Hamiltonian reads

$$\hat{H} = \int d\mathbf{r}dz\hat{\Psi}^\dagger(\mathbf{r}, z)\left[\frac{1}{2m^*}(\hat{p} - \frac{e}{c}\hat{A}(\mathbf{r}, z))^2 + V_{2D}(\mathbf{r}, z) + V_{et}(\mathbf{r}, z)\right]\hat{\Psi}(\mathbf{r}, z) + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'dzV(|\mathbf{r}-\mathbf{r}'|, z - z') \times \hat{\Psi}^\dagger(\mathbf{r}, z)\hat{\Psi}^\dagger(\mathbf{r}', z')\hat{\Psi}(\mathbf{r}', z')\hat{\Psi}(\mathbf{r}, z) \tag{68}$$

where V_{et} is the emitter potential, and the electron creation operator can be expanded to

$$\hat{\Psi}^\dagger(\mathbf{x}) = \hat{\Psi}^\dagger_{2D}(\mathbf{x}) + \hat{\Psi}^\dagger_{et}(\mathbf{x}) = \frac{1}{\sqrt{S}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\psi(z)\Psi^\dagger_{\mathbf{k}} + \sum_i \phi_i(\mathbf{x})\hat{\phi}^\dagger_i, \tag{69}$$

here $\hat{\phi}^\dagger_i$ is the i^{th} creation operator of emitter. Note that $\phi_i(\mathbf{x})$ is well localized, one reasonable assumption is its total electron number is conserved when interacting with SPPs, and we can only keep the emitter electron conserved terms in the total Hamiltonian

$$\hat{H} = \hat{H}_{et} + \hat{H}_{ee} + \hat{H}_{2D}, \tag{70}$$

where the last term $\hat{H}_{2D}$ is the Hamiltonian of 2D electron gas interacting with light defined previously, $\hat{H}_{et}$ is the emitter Hamiltonian under vector field $\hat{\mathbf{A}}(\mathbf{x})$ (note that emitter SPPs can interact through this photon field.). The last term $\hat{H}_{ee}$ is electron electron interaction Hamiltonian, generally, it can be expressed as the emitter electron interacting with fields generated by the 2D electron gas. Now

$$\hat{H}_{et} = \int \hat{\Psi}^\dagger_{et}(\mathbf{x})\left[\frac{1}{2m^*}(\hat{p} - \frac{e}{c}\hat{A}(\mathbf{x}))^2 + V_{et}(\mathbf{x})\right]\hat{\Psi}(\mathbf{x})d\mathbf{x}$$
$$\hat{H}_{ee} = \int\int \hat{\Psi}^\dagger_{et}(\mathbf{x})\hat{V}_e(\mathbf{x}, \mathbf{x}')\hat{\Psi}_{et}(\mathbf{x}')d\mathbf{x}d\mathbf{x}'. \tag{71}$$

The effective potential $\hat{V}_e(\mathbf{x}, \mathbf{x}')$ contains two parts, one originates from the direct Coulomb interaction $\delta(\mathbf{x}-\mathbf{x}')\hat{V}_{ec}(\mathbf{x})$, and the other one comes from electron exchanging effect $\hat{V}_{ex}(\mathbf{x}, \mathbf{x}')$, where

$$\hat{V}_{ec}(\mathbf{x}) = \int V(|\mathbf{x}-\mathbf{x}'|)\hat{\Psi}^\dagger_{2D}(\mathbf{x}')\hat{\Psi}_{2D}(\mathbf{x}')d\mathbf{x}' \tag{72}$$

$$\hat{V}_{ex}(\mathbf{x}, \mathbf{x}') = -V(|\mathbf{x}-\mathbf{x}'|)\hat{\Psi}^\dagger_{2D}(\mathbf{x}')\hat{\Psi}_{2D}(\mathbf{x}). \tag{73}$$

For the direct Coulomb potential Eq. (72), we have

$$\hat{V}_{ec}(\mathbf{x}) = \int V(|\mathbf{x}-\mathbf{x}'|)\hat{\Psi}^\dagger_{2D}(\mathbf{x}')\hat{\Psi}_{2D}(\mathbf{x}')d\mathbf{x}' = \frac{e^2}{S}\sum_{ss'\mathbf{qk}}\int \frac{e^{-i\mathbf{q}\cdot\mathbf{r}'}}{\sqrt{|\mathbf{r}-\mathbf{r}'|^2 + (z-z')^2}}|\psi(z')|^2\langle s\mathbf{k}+\mathbf{q}|s'\mathbf{k}\rangle\hat{\gamma}^\dagger_{\mathbf{k+q}s}\hat{\gamma}_{\mathbf{k}s'}d\mathbf{r}'dz' \tag{74}$$

The function $\zeta(q, z)$ is defined by

$$e^{-\zeta(q,z)q|z|} = \int e^{-q|z-z'|}|\psi(z')|^2dz', \tag{75}$$

note that the typical momentum we've considered in SPP is in the range $10^2 - 10^4$ cm^{-1}, and the envelop function width $b_s \sim 1$ nm, one immediately find $qb_s \ll 1$ and $\zeta(q, z) \approx 1$. As an example, consider the Gaussian type envelop function $b_s^{-1/2}(2\pi)^{-1/4}\exp(-z^2/4b_s^2)$, we have

$$e^{-\zeta(q,z)q|z|} = \frac{1}{\sqrt{2\pi}b_s}\int e^{-q|z-z'|}e^{-\frac{z'^2}{2b_s^2}}dz'$$
$$=\frac{1}{2}e^{b_sq(\frac{b_sq}{2}-\frac{z}{b_s})}\text{Erfc}\left[\frac{1}{\sqrt{2}}(b_sq - \frac{z}{b_s})\right] + \frac{1}{2}e^{b_sq(\frac{b_sq}{2}+\frac{z}{b_s})}\text{Erfc}\left[\frac{1}{\sqrt{2}}(b_sq + \frac{z}{b_s})\right]$$
$$=e^{-q|z|}\{1 - (\frac{z-|z|}{b_s} - \frac{z}{b_s}\text{Erfc}\left[\frac{z}{\sqrt{2}b_s}\right] + \sqrt{\frac{2}{\pi}}e^{-\frac{z^2}{2b_s^2}})b_sq + O(b_s^2q^2)\} \tag{76}$$

Note that the maximum value of $\frac{z-|z|}{b_s}-\frac{z}{b_s}\mathrm{Erfc}[\frac{z}{\sqrt{2}b_s}]+\sqrt{\frac{2}{\pi}}e^{-\frac{z^2}{2b_s^2}}$ is $\sqrt{2/\pi}$ when $z=0$, and it quickly decreases to zero when $z/b_s>5$. So, the error of replacing $e^{-\zeta(q,z)q|z|}$ with $e^{-q|z|}$ is smaller than $\sqrt{2/\pi}b_s q$, which means we can safely ignore $\zeta(q,z)$ in our formulas.

It should be emphasized here that, we want to derive the interaction between emitter and SPPs, instead of the interaction between emitter and 2D electron gas. To do so, we can transfer to momentum space and utilize the inverse transformation, and only keep the contribution from SPPs in $\hat{V}_e(\mathbf{x},\mathbf{x}')$. The inverse transformation gives

$$\begin{aligned}\hat{V}^{spp}_{ec}(\mathbf{x}) =&\frac{2\pi e^2}{\sqrt{2}S}\sum_{ss'\mathbf{qk}}\frac{e^{-i\mathbf{q}\cdot\mathbf{r}-qz}}{q}\Delta n^{ss'}_{\mathbf{kq}}\langle s\mathbf{k}+\mathbf{q}|s'\mathbf{k}\rangle[(\Phi_{ss'\mathbf{kq}})^*\hat{Q}^\dagger_q-\Phi_{s's\mathbf{k}+\mathbf{q},-\mathbf{q}}\hat{Q}_{-\mathbf{q}}]\\ =&\frac{1}{\sqrt{2}S}\sum_{ss'\mathbf{qk}}e^{-i\mathbf{q}\cdot\mathbf{r}-qz}V(q)\Delta n^{ss'}_{\mathbf{kq}}\langle s\mathbf{k}+\mathbf{q}|s'\mathbf{k}\rangle\Phi^*_{ss'\mathbf{kq}}\hat{Q}^\dagger_q+h.c.\\ =&\sum_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{r}-qz}L^{\rho*}_{\mathbf{q}}\hat{Q}^\dagger_{\mathbf{q}}+h.c.\end{aligned}\tag{77}$$

the superscript ^{spp} means only considering the contribution of SPPs. In dipole approximation, it is convenient to study the interaction through electric field, and the static part comes from $\hat{V}^{spp}_{ec}$ reads

$$\hat{\mathbf{E}}^s(\mathbf{x})=\frac{1}{e}\nabla\hat{V}^{spp}_c(\mathbf{x})=\frac{1}{e}\sum_{\mathbf{q}}(-i\mathbf{q},-q)e^{-i\mathbf{q}\cdot\mathbf{r}-qz}L^*_q\hat{Q}^\dagger_q+h.c.\tag{78}$$

For the radiation part, the contribution of SPP to vector field reads

$$\begin{aligned}\hat{a}_{\lambda,-\mathbf{q},-p_z}&=-\Phi^{a*}_{\lambda,p_z}\hat{Q}^\dagger_{\mathbf{q}}+\Phi^c_{\lambda,-p_z}\hat{Q}_{-\mathbf{q}}\\ \hat{a}^\dagger_{\lambda\mathbf{q}p_z}&=\Phi^{c*}_{\lambda,p_z}\hat{Q}^\dagger_{\mathbf{q}}-\Phi^a_{\lambda,-p_z}\hat{Q}_{-\mathbf{q}}\end{aligned}\tag{79}$$

Now, the vector potential

$$\begin{aligned}\hat{A}_x(\mathbf{r},z)=&\sum_{\mathbf{q}p_z}(\frac{2\pi\hbar c^2}{\nu\omega_{\mathbf{q}p_z}})^{1/2}e^{i\mathbf{q}\cdot\mathbf{r}+ip_zz}\left[\frac{p_z\cos\theta_{\mathbf{q}}}{\sqrt{q^2+p_z^2}}(\hat{a}_{2\mathbf{q}p_z}+\hat{a}^\dagger_{2,-\mathbf{q},-p_z})+\sin\theta_{\mathbf{q}}(\hat{a}_{1\mathbf{q}p_z}-\hat{a}^\dagger_{1,-\mathbf{q},-p_z})\right]\\ =&-\sum_{\mathbf{q}p_z}(\frac{2\pi\hbar c^2}{\nu\omega_{\mathbf{q}p_z}})^{1/2}e^{-i\mathbf{q}\cdot\mathbf{r}-ip_zz}\left[\frac{-p_z\cos\theta_{\mathbf{q}}}{\sqrt{q^2+p_z^2}}(\hat{a}_{2,-\mathbf{q},-p_z}+\hat{a}^\dagger_{2\mathbf{q}p_z})+\sin\theta_{\mathbf{q}}(\hat{a}_{1,-\mathbf{q},-p_z}-\hat{a}^\dagger_{1\mathbf{q}p_z})\right]\\ =&-\frac{c}{eL_z}\sum_{\mathbf{q}p_z}e^{-i\mathbf{q}\cdot\mathbf{r}-ip_zz}\left[\frac{p_z^2\cos\theta_{\mathbf{q}}}{p_z^2+q^2}\frac{\Omega_{\mathbf{q}}}{\omega_{\mathbf{q}p_z}}\left(\frac{1}{\Omega_{\mathbf{q}}+\omega_{\mathbf{q}p_z}}-\frac{1}{\Omega_{\mathbf{q}}-\omega_{\mathbf{q}p_z}}\right)(L^{\rho*}_{\mathbf{q}}\hat{Q}^\dagger_{\mathbf{q}}-L^\rho_{-\mathbf{q}}\hat{Q}_{-\mathbf{q}})\right.\\ &\left.-\frac{c^2q'}{\omega_{\mathbf{q}p_z}}\sin\theta_{\mathbf{q}}\left(\frac{1}{\Omega_{\mathbf{q}}+\omega_{\mathbf{q}p_z}}-\frac{1}{\Omega_{\mathbf{q}}-\omega_{\mathbf{q}p_z}}\right)(L^{y*}_{\mathbf{q}}\hat{Q}^\dagger_{\mathbf{q}}-L^y_{-\mathbf{q}}\hat{Q}_{-\mathbf{q}})\right]\\ =&-\frac{c}{e}\sum_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{r}}\left[L^{\rho*}_{\mathbf{q}}\cos\theta_{\mathbf{q}}\left(\frac{qe^{-qz}}{\Omega_{\mathbf{q}}}-\frac{q'e^{-q'z}}{\Omega_{\mathbf{q}}}\right)-L^{y*}_{\mathbf{q}}e^{-q'z}\sin\theta_{\mathbf{q}}\right]\hat{Q}^\dagger_{\mathbf{q}}+h.c.\end{aligned}\tag{80}$$

and

$$\begin{aligned}\hat{A}_y(\mathbf{r},z)=&\sum_{\mathbf{q}p_z}\left(\frac{2\pi\hbar c^2}{\nu\omega_{\mathbf{q}p_z}}\right)^{1/2}e^{i\mathbf{q}\cdot\mathbf{r}+iq_zz}\left[\frac{q_z\sin\theta_{\mathbf{q}}}{\sqrt{q^2+q_z^2}}(\hat{a}_{2\mathbf{q}q_z}+\hat{a}^\dagger_{2,-\mathbf{q},-q_z})-\cos\theta_{\mathbf{q}}(\hat{a}_{1\mathbf{q}q_z}-\hat{a}^\dagger_{1,-\mathbf{q},-q_z})\right]\\ =&\sum_{\mathbf{q}q_z}\left(\frac{2\pi\hbar c^2}{\nu\omega_{\mathbf{q}q_z}}\right)^{1/2}e^{-i\mathbf{q}\cdot\mathbf{r}-iq_zz}\left[\frac{q_z\sin\theta_{\mathbf{q}}}{\sqrt{q^2+q_z^2}}(\hat{a}_{2,-\mathbf{q},-q_z}+\hat{a}^\dagger_{2\mathbf{q}q_z})+\cos\theta_{\mathbf{q}}(\hat{a}_{1,-\mathbf{q},-q_z}-\hat{a}^\dagger_{1\mathbf{q}q_z})\right]\\ =&-\frac{c}{e}\sum_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{r}}\left[L^{\rho*}_{\mathbf{q}}\sin\theta_{\mathbf{q}}\left(\frac{qe^{-qz}}{\omega_{\mathbf{q}}}-\frac{q'e^{-q'z}}{\omega_{\mathbf{q}}}\right)+L^{y*}_{\mathbf{q}}e^{-q'z}\cos\theta_{\mathbf{q}}\right]\hat{Q}^\dagger_{\mathbf{q}}+h.c.\end{aligned}\tag{81}$$

the z direction

$$\hat{A}_z(\mathbf{r},z) = -\sum_{\mathbf{q}q_z}\left(\frac{2\pi\hbar c^2}{\nu\omega_{\mathbf{q}q_z}}\right)^{1/2} e^{-i\mathbf{q}\cdot\mathbf{r}-iq_z z}\frac{q}{\sqrt{q^2+q_z^2}}(\hat{a}_{2,-\mathbf{q},-q_z}+\hat{a}^\dagger_{2\mathbf{q}q_z})$$
$$=\frac{c}{eL_z}\sum_{\mathbf{q}q_z} e^{-i\mathbf{q}\cdot\mathbf{r}-iq_z z}\frac{qq_z}{q_z^2+q^2}\frac{\omega_\mathbf{q}}{\omega_{\mathbf{q}q_z}}\left(\frac{1}{\omega_\mathbf{q}+\omega_{\mathbf{q}q_z}}-\frac{1}{\omega_\mathbf{q}-\omega_{\mathbf{q}q_z}}\right)(L^{\rho*}_\mathbf{q}\hat{Q}^\dagger_\mathbf{q}-L^\rho_{-\mathbf{q}}\hat{Q}_{-\mathbf{q}})$$
$$=\frac{ic}{e}\sum_\mathbf{q} e^{-i\mathbf{q}\cdot\mathbf{r}}\frac{L^{\rho*}_\mathbf{q}q}{\omega_\mathbf{q}}(e^{-qz}-e^{-q'z})\hat{Q}^\dagger_\mathbf{q}+h.c. \tag{82}$$

Now, the electric field from vector field reads $\hat{\mathbf{E}}^r(\mathbf{r},z) = -\partial_t\hat{\mathbf{A}}(\mathbf{r},z)/c$ and

$$\hat{\mathbf{E}}^r_x(\mathbf{r},z) = \frac{i}{e}\sum_\mathbf{q} e^{-i\mathbf{q}\cdot\mathbf{r}}[L^{\rho*}_\mathbf{q}\cos\theta_\mathbf{q}(qe^{-qz}-q'e^{-q'z})-\Omega_\mathbf{q}L^{y*}_\mathbf{q}e^{-q'z}\sin\theta_\mathbf{q}]\hat{Q}^\dagger_\mathbf{q}+h.c.$$
$$\hat{\mathbf{E}}^r_y(\mathbf{r},z) = \frac{i}{e}\sum_\mathbf{q} e^{-i\mathbf{q}\cdot\mathbf{r}}[L^{\rho*}_\mathbf{q}\sin\theta_\mathbf{q}(qe^{-qz}-q'e^{-q'z})+\Omega_\mathbf{q}L^{y*}_\mathbf{q}e^{-q'z}\cos\theta_\mathbf{q}]\hat{Q}^\dagger_\mathbf{q}+h.c.$$
$$\hat{\mathbf{E}}^r_z(\mathbf{r},z) = \frac{1}{e}\sum_\mathbf{q} e^{-i\mathbf{q}\cdot\mathbf{r}}L^{\rho*}_\mathbf{q}q(e^{-qz}-e^{-q'z})\hat{Q}^\dagger_\mathbf{q}+h.c. \tag{83}$$

where $\tilde{\mathbf{q}}$ is a in-plane vector perpendicular to $\mathbf{q}$ ($\theta_{\tilde{\mathbf{q}}}=\theta_\mathbf{q}+\pi/2$, illustrated in Figure 1c in the main text), and $\mathbf{q}'$ means the magnitude of $\mathbf{q}$ reduced from q to $q'=\sqrt{q^2-\Omega_\mathbf{q}^2/c^2}$. Finally, the total electric field reads

$$\hat{E}(\mathbf{x}) = \hat{E}^s(\mathbf{x})+\hat{\mathbf{E}}^r(\mathbf{x}) = -\frac{1}{e}\sum_\mathbf{q} e^{-i\mathbf{q}\cdot\mathbf{r}-q'z}[(i\mathbf{q}',q)L^{\rho*}_\mathbf{q}+\Omega_\mathbf{q}\frac{\tilde{\mathbf{q}}}{q}L^{y*}_\mathbf{q}]\hat{Q}^\dagger_\mathbf{q}+h.c.. \tag{84}$$

Now, the interaction Hamiltonian through electric field $\hat{H}^f_{int} = -\int\hat{\mu}(\mathbf{x})\cdot\hat{\mathbf{E}}(\mathbf{x})d\mathbf{x}$ reads

$$\hat{H}^f_{int} = e\int\hat{\Psi}^\dagger_{et}(\mathbf{x})[\mathbf{x}\cdot\hat{\mathbf{E}}(\mathbf{x})]\hat{\Psi}_{et}(\mathbf{x})d\mathbf{x}$$
$$\approx\sum_{ij\mathbf{q}} g^{ij}_f(\mathbf{q})\hat{\phi}^\dagger_i\hat{\phi}_j\hat{Q}^\dagger_\mathbf{q}+h.c., \tag{85}$$

with the interaction strength

$$g^{ij}_f(\mathbf{q}) = -i(\mathbf{q}'\cdot\mathbf{d}^{||}_{ij}+qd^\perp_{ij})L^{\rho*}_\mathbf{q}-\frac{\Omega_\mathbf{q}}{q}\tilde{\mathbf{q}}\cdot\mathbf{d}^{||}_{ij}L^{y*}_\mathbf{q}, \tag{86}$$

where $\mathbf{d}_{ij} = \int\phi^*_i(\mathbf{x})\mathbf{x}\phi_j(\mathbf{x})d\mathbf{x}$ is the transition dipole moment, the superscript $^{||},^{\perp}$ means the parallel and vertical parts.

The Hamiltonian originates from exchanging effect reads

$$\hat{H}_{ex} = \int d\mathbf{x}d\mathbf{x}'\hat{\phi}^\dagger(\mathbf{x})\hat{\phi}(\mathbf{x}')V(|\mathbf{x}-\mathbf{x}'|)\hat{\Psi}^\dagger(\mathbf{x}')\hat{\Psi}(\mathbf{x})$$
$$=\frac{1}{S}\sum_\mathbf{q}\sum_{ij\mathbf{k}}\hat{\Psi}^\dagger_{\mathbf{k}+\mathbf{q}}\hat{\Psi}_\mathbf{k}\hat{\phi}^\dagger_i\hat{\phi}_j\int d\mathbf{r}d\mathbf{r}'dzdz'\psi(z)\psi^*(z')\phi^*_i(\mathbf{r})\phi_j(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}'}V(|\mathbf{r}-\mathbf{r}'|,z-z')$$
$$=\sum_{ij\mathbf{q}} g^{ex}_{ij}(\mathbf{q})\hat{\phi}^\dagger_i\hat{\phi}_j\hat{Q}^\dagger_\mathbf{q}+h.c. \tag{87}$$

where the interaction strength reads

$$g^{ex}_{ij}(\mathbf{q}) = \frac{1}{S}\sum_{\mathbf{k}ss'}\Delta n^{ss'}_{\mathbf{k}\mathbf{q}}\langle s\mathbf{k}+\mathbf{q}|s'\mathbf{k}\rangle(\Phi_{ss'\mathbf{k}\mathbf{q}})^*\int d\mathbf{r}d\mathbf{r}'dzdz'\psi(z)\psi^*(z')\phi^*_i(\mathbf{r})\phi_j(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}'}V(|\mathbf{r}-\mathbf{r}'|,z-z')$$
$$=\frac{1}{S}\sum_{ss'\mathbf{k}}\frac{\Delta n^{ss'}_{\mathbf{k}\mathbf{q}}}{\hbar\omega_\mathbf{q}-\Delta\xi^{ss'}_{\mathbf{k}\mathbf{q}}}[L^\rho_\mathbf{q}|W^{ss'}_{\mathbf{k}\mathbf{q}}|^2+L^\rho_\mathbf{q}\frac{q-q'}{\Omega_\mathbf{q}}(W^{ss'}_{\mathbf{k}\mathbf{q}})^*Y^{ss'}_{\mathbf{k}\mathbf{q}}v_f+L^y_\mathbf{q}(W^{ss'}_{\mathbf{k}\mathbf{q}})^*X^{ss'}_{\mathbf{k}\mathbf{q}}v_f]^*$$
$$\times\int d\mathbf{r}d\mathbf{r}'dzdz'\psi(z)\psi^*(z')\phi^*_i(\mathbf{r})\phi_j(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}'}V(|\mathbf{r}-\mathbf{r}'|,z-z') \tag{88}$$

For convenience, we have defined

$$\begin{aligned} W_{\mathbf{kq}}^{ss'} &= \langle s\mathbf{k}+\mathbf{q}|s'\mathbf{k}\rangle, \\ X_{\mathbf{kq}}^{ss'} &= \langle s\mathbf{k}+\mathbf{q}|\sigma_x|s'\mathbf{k}\rangle, \\ Y_{\mathbf{kq}}^{ss'} &= \langle s\mathbf{k}+\mathbf{q}|\sigma_y|s'\mathbf{k}\rangle \end{aligned} \tag{89}$$

The exchange strength is quite complicated and should be calculated numerically.

$$\begin{aligned} g_{ij}^{ex}(\mathbf{q}) = &\frac{1}{S}\sum_{ss'\mathbf{k}}\frac{\Delta n_{\mathbf{kq}}^{ss'}}{\hbar\omega_{\mathbf{q}}-\Delta\xi_{\mathbf{kq}}^{ss'}}[L_{\mathbf{q}}^{\rho}|W_{\mathbf{kq}}^{ss'}|^2+L_{\mathbf{q}}^{\rho}\frac{q-q'}{\Omega_{\mathbf{q}}}(W_{\mathbf{kq}}^{ss'})^*Y_{\mathbf{kq}}^{ss'}v_f+L_{\mathbf{q}}^{y}(W_{\mathbf{kq}}^{ss'})^*X_{\mathbf{kq}}^{ss'}v_f]^* \\ &\times\int d\mathbf{r}d\mathbf{r}'dzdz'\psi(z)\psi^*(z')\phi_i^*(\mathbf{r},z)\phi_j(\mathbf{r}',z')e^{i\mathbf{k}\cdot\mathbf{r}-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}'}V(|\mathbf{r}-\mathbf{r}'|,z-z') \\ =&\frac{1}{S}\sum_{ss'\mathbf{k}}\frac{\Delta n_{\mathbf{kq}}^{ss'}}{\hbar\omega_{\mathbf{q}}-\Delta\xi_{\mathbf{kq}}^{ss'}}[L_{\mathbf{q}}^{\rho}|W_{\mathbf{kq}}^{ss'}|^2+L_{\mathbf{q}}^{\rho}\frac{q-q'}{\Omega_{\mathbf{q}}}(W_{\mathbf{kq}}^{ss'})^*Y_{\mathbf{kq}}^{ss'}v_f+L_{\mathbf{q}}^{y}(W_{\mathbf{kq}}^{ss'})^*X_{\mathbf{kq}}^{ss'}v_f]^* \\ &\times\int d\mathbf{r}d\mathbf{r}'dzdz'\psi(z)\psi^*(z')\phi_i^*(\mathbf{r},z)\phi_j(\mathbf{r}',z')e^{i\mathbf{k}\cdot\mathbf{r}-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}'}\frac{e^2}{2\pi^2}\int d\mathbf{p}dp_z\frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')+ip_z(z-z')}}{p^2+p_z^2} \\ =&\frac{\lambda_f e^2 L_{\mathbf{q}}^*}{4\pi^2 S}\sum_{ss'\mathbf{k}}\frac{\Delta n_{\mathbf{kq}}^{ss'}}{\hbar\omega_{\mathbf{q}}-\Delta\xi_{\mathbf{kq}}^{ss'}}[|W_{\mathbf{kq}}^{ss'}|^2+\frac{q-q'}{\Omega_{\mathbf{q}}}(W_{\mathbf{kq}}^{ss'})^*Y_{\mathbf{kq}}^{ss'}v_f+\frac{1-V_{ql}\Pi_{j_xj_x}}{V_q\Pi_{j_y\rho}}(W_{\mathbf{kq}}^{ss'})^*X_{\mathbf{kq}}^{ss'}v_f]^* \\ &\times\frac{k_f}{\pi}\int d\mathbf{r}d\mathbf{r}'dzdz'\psi(z)\psi^*(z')\phi_i^*(\mathbf{r},z)\phi_j(\mathbf{r}',z')\frac{e^{i(\mathbf{k}+\mathbf{p})\cdot\mathbf{r}-i(\mathbf{k}+\mathbf{p}+\mathbf{q})\cdot\mathbf{r}'}e^{ip_z(z-z')}}{p^2+p_z^2} \end{aligned} \tag{90}$$

Now, let's define

$$\eta_{\mathbf{kq}}^{ij} = \frac{k_f}{\pi}\int d\mathbf{r}d\mathbf{r}'dzdz'\psi(z)\psi^*(z')\phi_i^*(\mathbf{r},z)\phi_j(\mathbf{r}',z')\frac{e^{i(\mathbf{k}+\mathbf{p})\cdot\mathbf{r}-i(\mathbf{k}+\mathbf{p}+\mathbf{q})\cdot\mathbf{r}'}e^{ip_z(z-z')}}{p^2+p_z^2}, \tag{91}$$

$$\eta_{\mathbf{q}}^{ij} = \frac{V_q}{S}\sum_{ss'\mathbf{k}}\frac{\eta_{\mathbf{kq}}^{ij}\Delta n_{\mathbf{kq}}^{ss'}}{\hbar\omega_{\mathbf{q}}-\Delta\xi_{\mathbf{kq}}^{ss'}}[|W_{\mathbf{kq}}^{ss'}|^2+\frac{q-q'}{\Omega_{\mathbf{q}}}(W_{\mathbf{kq}}^{ss'})^*Y_{\mathbf{kq}}^{ss'}v_f+\frac{1-V_{ql}\Pi_{j_xj_x}}{V_q\Pi_{j_y\rho}}(W_{\mathbf{kq}}^{ss'})^*X_{\mathbf{kq}}^{ss'}v_f]^* \tag{92}$$

We can express the exchange interaction strength in a simpler form

$$g_{ij}^{ex}(\mathbf{q}) = \frac{q\lambda_f L_{\mathbf{q}}^*}{8\pi^3}\eta_{\mathbf{q}}^{ij} \tag{93}$$

Suppose the envelop function $\psi(z)$ is a Gaussian function $b_s^{-1/2}(2\pi)^{-1/4}\exp(-z^2/4b_s^2)$, and the emitters are 2D hydrogen like atoms with envelop function $b_e^{-1/2}(2\pi)^{-1/4}\exp(-(z-d)^2/4b_e^2)$, we have

$$\begin{aligned} \eta_{\mathbf{kq}}^{ij} = &\frac{k_f}{\pi}\int dp_z d\mathbf{r}d\mathbf{r}'dzdz'\psi(z)\psi^*(z')\phi_i^*(\mathbf{r},z)\phi_j(\mathbf{r}',z')\frac{e^{i\mathbf{p}\cdot\mathbf{r}-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}'}e^{ip_z(z-z')}}{p^2+k^2-2pk\cos(\theta_{\mathbf{p}}-\theta_{\mathbf{k}})+p_z^2} \\ =&\frac{k_f e^{-d^2/2b_e^2}}{2\pi^2 b_s b_e}\int d\mathbf{p}dp_z d\mathbf{r}d\mathbf{r}'\phi_i^*(\mathbf{r})\phi_j(\mathbf{r}')\frac{e^{i\mathbf{p}\cdot\mathbf{r}-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}'}}{p^2+k^2-2pk\cos(\theta_{\mathbf{p}}-\theta_{\mathbf{k}})+p_z^2}\int e^{-\frac{1}{4}(\frac{1}{b_s^2}+\frac{1}{b_e^2})z^2+(\frac{d}{2b_e^2}+ip_z)z}e^{-\frac{1}{4}(\frac{1}{b_s^2}+\frac{1}{b_e^2})z'^2+(\frac{d}{2b_e^2}-ip_z)z'}dzdz' \\ =&\frac{k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{2\pi^2(b_s^2+b_e^2)}\int prr'dp_z dp dr dr' d\theta_{\mathbf{p}}d\theta_{\mathbf{r}}d\theta_{\mathbf{r}'}e^{-im_i\theta_{\mathbf{r}}}\phi_i^*(r)e^{im_j\theta_{\mathbf{r}'}}\phi_j(r')\frac{e^{ipr\cos(\theta_{\mathbf{p}}-\theta_{\mathbf{r}})-i|\mathbf{p}+\mathbf{q}|r'\cos(\theta_{\mathbf{p}+\mathbf{q}}-\theta_{\mathbf{r}'})}}{p^2+k^2-2pk\cos(\theta_{\mathbf{p}}-\theta_{\mathbf{k}})+p_z^2}e^{-2\frac{b_s^2b_e^2}{b_s^2+b_e^2}p_z^2} \\ =&\frac{k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{2\pi^2(b_s^2+b_e^2)}\int prr'dp_z dp dr dr' d\theta'_{\mathbf{p}}d\theta'_{\mathbf{r}}d\theta'_{\mathbf{r}'}\phi_i^*(\mathbf{r})\phi_j(\mathbf{r}')\frac{e^{ipr\cos\theta'_{\mathbf{r}}-i|\mathbf{p}+\mathbf{q}|r'\cos\theta'_{\mathbf{r}'}-im_i(\theta'_{\mathbf{r}}+\theta'_{\mathbf{p}}+\theta_{\mathbf{k}})+im_j(\theta'_{\mathbf{r}'}+\theta_{\mathbf{p}+\mathbf{q}})}}{p^2+k^2-2pk\cos\theta'_{\mathbf{p}}+p_z^2}e^{-2\frac{b_s^2b_e^2}{b_s^2+b_e^2}p_z^2} \\ \approx&\frac{k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{2\pi^2(b_s^2+b_e^2)}\int prr'dp_z dp dr dr' d\theta'_{\mathbf{p}}d\theta'_{\mathbf{r}}d\theta'_{\mathbf{r}'}\phi_i^*(\mathbf{r})\phi_j(\mathbf{r}')\frac{e^{ipr\cos\theta'_{\mathbf{r}}-ipr'\cos\theta'_{\mathbf{r}'}-im_i(\theta'_{\mathbf{r}}+\theta'_{\mathbf{p}}+\theta_{\mathbf{k}})+im_j(\theta'_{\mathbf{r}'}+\theta'_{\mathbf{p}}+\theta_{\mathbf{k}})}}{p^2+k^2-2pk\cos\theta'_{\mathbf{p}}+p_z^2}e^{-2\frac{b_s^2b_e^2}{b_s^2+b_e^2}p_z^2} \end{aligned} \tag{94}$$

where we have applied the limit $q/k_f \to 0$ (expanding $e^{i\mathbf{q}\cdot\mathbf{r}'}$ and only keep the constant term), when $m_i = m_j = 0$, we have $\phi_{i,j}(\mathbf{r}) = 1/\sqrt{2\pi}\phi_{i,j}(r)$, and can get a simple expression

$$\eta^{ij}_{\mathbf{kq}} = \frac{k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{4\pi^3(b_s^2+b_e^2)} \int prr'dp_z dpdrdr'd\theta'_{\mathbf{p}} d\theta'_{\mathbf{r}} d\theta'_{\mathbf{r}'} \phi_i^*(r)\phi_j(r') \frac{e^{ipr\cos\theta'_{\mathbf{r}} - ipr'\cos\theta'_{\mathbf{r}'}}}{p^2+k^2-2pk\cos\theta'_{\mathbf{p}}+p_z^2} e^{-2\frac{b_s^2 b_e^2}{b_s^2+b_e^2}p_z^2}$$

$$= \frac{k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{\pi(b_s^2+b_e^2)} \int prr'dp_z dpdrdr'd\theta'_{\mathbf{p}} \phi_i^*(r)\phi_j(r') \frac{J_0(pr)J_0(pr')}{p^2+k^2-2pk\cos\theta'_{\mathbf{p}}+p_z^2} e^{-2\frac{b_s^2 b_e^2}{b_s^2+b_e^2}p_z^2}$$

$$= \frac{k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{(b_s^2+b_e^2)} \int prr'dpdrdr'd\theta'_{\mathbf{p}} \phi_i^*(r)\phi_j(r') \frac{J_0(pr)J_0(pr')}{\sqrt{p^2+k^2-2pk\cos\theta'_{\mathbf{p}}}} e^{2\frac{b_s^2 b_e^2}{b_s^2+b_e^2}(p^2+k^2-2pk\cos\theta'_{\mathbf{p}})}$$

$$\times \operatorname{Erfc}\left(\sqrt{2}\frac{b_s b_e}{\sqrt{b_s^2+b_e^2}}\sqrt{p^2+k^2-2pk\cos\theta'_{\mathbf{p}}}\right)$$

$$= \frac{k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{(b_s^2+b_e^2)} \int prr'dpdrdr'd\theta'_{\mathbf{p}} \phi_i^*(r)\phi_j(r') \frac{J_0(pr)J_0(pr')}{\sqrt{p^2+k^2-2pk\cos\theta'_{\mathbf{p}}}}$$

$$\times \left[1 - 2\sqrt{2/\pi}\frac{b_s b_e}{\sqrt{b_s^2+b_e^2}}\sqrt{p^2+k^2-2pk\cos\theta'_{\mathbf{p}}} + O\left(\frac{b_s^2 b_e^2}{b_s^2+b_e^2}(p^2+k^2-2pk\cos\theta'_{\mathbf{p}})\right)\right]$$

$$\approx \frac{k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{(b_s^2+b_e^2)} \int prr'dpdrdr'd\theta'_{\mathbf{p}} \phi_i^*(r)\phi_j(r') \frac{J_0(pr)J_0(pr')}{\sqrt{p^2+k^2-2pk\cos\theta'_{\mathbf{p}}}}$$

$$= \frac{2k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{b_s^2+b_e^2} \int prr'dpdrdr' \phi_i^*(r)\phi_j(r') J_0(pr)J_0(pr') \left[\frac{K(-4kp/(k-p)^2)}{|k-p|} + \frac{K(4kp/(k+p)^2)}{k+p}\right] \tag{95}$$

which means $\eta^{ij}_{\mathbf{kq}}$ is independent with the direction of $\mathbf{k}$, and from now on, we write it as $\eta^{ij}_{k\mathbf{q}}$. Note that $\Delta n^{ss'}_{\mathbf{kq}}$ is none zero only when $k \approx k_f$ at the limit of $q/k_f \to 0$, so we have

$$\eta^{ij}_{\mathbf{q}} \approx \eta^{ij}_{k_f\mathbf{q}} \frac{V_q}{S} \sum_{ss'\mathbf{k}} \frac{\Delta n^{ss'}_{\mathbf{kq}}}{\hbar\omega_{\mathbf{q}} - \Delta\xi^{ss'}_{\mathbf{kq}}} \left[|W^{ss'}_{\mathbf{kq}}|^2 + \frac{q-q'}{\Omega_{\mathbf{q}}}(W^{ss'}_{\mathbf{kq}})^* Y^{ss'}_{\mathbf{kq}} v_f + \frac{1 - V_{ql}\Pi_{j_x j_x}}{V_q \Pi_{j_y \rho}}(W^{ss'}_{\mathbf{kq}})^* X^{ss'}_{\mathbf{kq}} v_f\right]^*$$
$$= \eta^{ij}_{k_f\mathbf{q}} \tag{96}$$

As examples, we calculate $\eta^{00,10}$

$$\eta^{00,10}_{k_f\mathbf{q}}$$
$$= \frac{2k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{b_s^2+b_e^2} \int prr'dpdrdr' \frac{16}{3\sqrt{3}a_0^2}\left(1 - \frac{4r'}{3a_0}\right) e^{-\frac{2r}{a_0} - \frac{2r'}{3a_0}} J_0(pr)J_0(pr') \left[\frac{K(-4k_f p/(k_f-p)^2)}{|k_f-p|} + \frac{K(4k_f p/(k_f+p)^2)}{k_f+p}\right]$$
$$= \frac{2k_f b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{b_s^2+b_e^2} \int pdp \frac{16}{3\sqrt{3}} \frac{108a_0^2\left(9a_0^2p^2-4\right)}{\left(a_0^2p^2+4\right)^{3/2}\left(9a_0^2p^2+4\right)^{5/2}} \left[\frac{K(-4k_f p/(k_f-p)^2)}{|k_f-p|} + \frac{K(4k_f p/(k_f+p)^2)}{k_f+p}\right]$$
$$= \frac{2b_s b_e e^{-d^2/2(b_s^2+b_e^2)}}{b_s^2+b_e^2} \int xdx \frac{16}{3\sqrt{3}} \frac{108a_0^2k_f^2\left(9a_0^2k_f^2x^2-4\right)}{\left(a_0^2k_f^2x^2+4\right)^{3/2}\left(9a_0^2k_f^2x^2+4\right)^{5/2}} \left[\frac{K(-4x/(1-x)^2)}{|1-x|} + \frac{K(4x/(1+x)^2)}{1+x}\right] \tag{97}$$

For $b_s = b_e$ and $d = 0$, we've plotted $\eta_{00,10}$ in Fig. 1 at the typical parameters $a_0 \in (1, 5)$ nm, $k_f \approx 3 \times 10^8$ m^{-1} (when $v_f \approx 5 \times 10^5$ m/s, $\mu \approx 100$ meV), we have $a_0 k_f \in (0.3, 1.5)$, and $\eta(k_f)_{00,10} \in (-0.87, 0.12)$.

Finally, we can calculate the exchange interaction to field interaction ratio

$$g^{ij}_{ex}(\mathbf{q})/g^{ij}_f(\mathbf{q}) = i\frac{\eta_{ij}(k_f)}{8\pi^3}\frac{q\lambda_f}{\mathbf{d}_{ij}\cdot\mathbf{q}'} \tag{98}$$

Note that

$$\mathbf{d}_{00,11} = \int rdrd\theta \frac{32r^2\cos^2\theta}{9\sqrt{6}\pi a_0^3} e^{-\frac{8r}{3a_0}} = a_0 \int dx \frac{32}{9\sqrt{6}} x^3 e^{-\frac{8}{3}x} = \frac{27}{64\sqrt{6}} a_0 \tag{99}$$

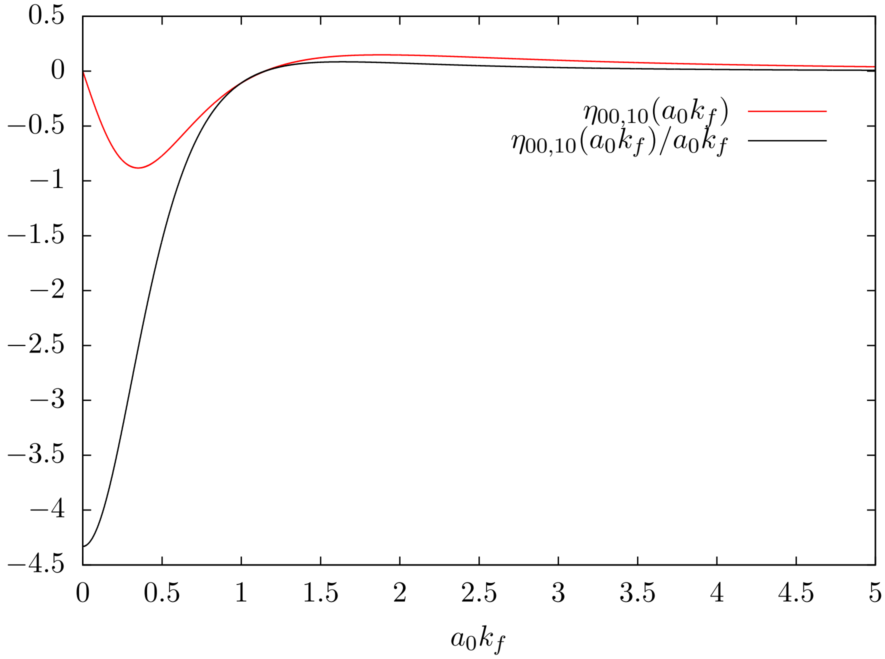

FIG. 1. (Color online) $\eta_{00,10}$ and $\eta_{00,10}/a_0k_f$ as functions of a_0k_f.

we can find

$$\frac{g_{ex}^{00,10}(\mathbf{q})}{g_f^{00,11}(\mathbf{q})} = i\frac{16\sqrt{6}}{27\pi^2}\frac{\eta_{k_f\mathbf{q}}^{00,10}}{a_0k_f}\frac{q}{q'} = i\frac{16\sqrt{6}}{27\pi^2}\frac{\eta_{k_f\mathbf{q}}^{00,10}}{a_0k_f}\frac{E_{||}^s}{E_{||}} \tag{100}$$