Controlled Manipulation with a Bose–Einstein Condensates N-Soliton Train under the Influence of Harmonic and Tilted Periodic Potentials *

ZONG Feng-De(宗丰德), ZHANG Jie-Fang(张解放)**
Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua 321004

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A model of the perturbed complex Toda chain (PCTC) to describe the dynamics of a Bose–Einstein condensate (BEC) N-soliton train trapped in an applied combined external potential consisting of both a weak harmonic and tilted periodic component is first developed. Using the developed theory, the BEC N-soliton train dynamics is shown to be well approximated by $4N$ coupled nonlinear differential equations, which describe the fundamental interactions in the system arising from the interplay of amplitude, velocity, centre-of-mass position, and phase. The simplified analytic theory allows for an efficient and convenient method for characterizing the BEC N-soliton train behaviour. It further gives the critical values of the strength of the potential for which one or more localized states can be extracted from a soliton train and demonstrates that the BEC N-soliton train can move selectively from one lattice site to another by simply manipulating the strength of the potential.

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Recently, Bose–Einstein condensate (BEC) solitons have attracted a great deal of interest both from the theoretical and the experimental point of view,[1,2] and have been used to study such diverse phenomena as phase coherence,[3−5] matter–wave diffraction,[6] quantum logic,[7−9] matter–wave transport,[10,11] matter–wave gratings, and pulsed matter–wave lasers.[12] Some dynamical issues related to a BEC N-soliton train, such as the dynamics of a train of matter–wave solitons confined to a parabolic trap, optical lattice, and tilted periodic potentials[13] have been investigated. However, so far the controlled manipulation dynamics of a BEC N-soliton train trapped in an applied combined external potential has not yet been analysed. In this Letter, we first generalize the model of the perturbed complex Toda chain (PCTC)[14−17] to a BEC N-soliton train, and further consider the dynamical evolution of a BEC N-soliton train as well as how to manipulate and control it subject to the influence of both harmonic and tilted periodic trapping potentials.[3,4,18] Our analysis characterizes the dynamic transition from trough to trough and further predicts when multiple lattice sites are stable. The analysis also suggests how to selectively move the BEC N-soliton train from one lattice site to another by simply manipulating the strength of the potential.

It has been well known that the macroscopic description of the attractive BEC wavefunction in cigar-shaped BEC trap geometries is governed by the nonlinear Schrödinger equation with external potential $V(x)$[19] (or Gross–Pitaevskii equation[20,21])

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2u=V(x)u, \tag{1}$$

where $u(x,t)$ is the normalized complex-valued mean-field variable, $V(x)$ is a real physical potential.

We will consider an applied combined external potential consisting of both a harmonic and tilted periodic component, thus the potential $V(x)$ in Eq. (1) is given by

$$V(x)=V_0x^2+V_1\cos[\Omega(x-\bar{x})]+V_2x, \tag{2}$$

where V_0, V_1, and V_2 measure the strengths of the harmonic, periodic, and tilted potentials, respectively. The parameter $\bar{x}$ measures the offset of a minimum of the periodic potential with respect to the minimum of the harmonic potential.

Note that although $V_0\sim 10^{-5}-10^{-3}$ is a small parameter in the range of realistic experimental conditions,[22,23] for any non-zero values of V_0 and V_2 in Eq. (2) the parabolic and tiled potentials tend to infinity as $|x|\to\infty$, and the external potential (2) is an unbounded operator. However, for a lattice of finite size $-L\le x\le L$ with L being the finite boundary value of x,[24] the operator (2) is always bounded. In the following, we consider the above weak potentials in the axial direction x and assume them as perturbations $iR[u]=V(x)u(x,t)$. With the effect of all these physical processes mentioned above, the BEC dynamics is described by the perturbed nonlinear Schrödinger equation (NLS)

$$i\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+|u|^2u=i\varepsilon R[u]. \tag{3}$$

We first generalize the PCTC model to a BEC with weak harmonic and tilted periodic external potentials in the adiabatic approximation. We concentrate on the perturbed NLS Eq. (3). By 'N-soliton train' we

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** To whom correspondence should be addressed. Email: jf_zhang@zjnu.cn)

mean a solution of the perturbed NLS equation fixed up by the initial condition

$$u(x,t=0)=\sum_{k=1}^{N}u_k^{1s}(x,t=0),$$
$$u_k^{1s}(x,t=0)=2\nu_k \mathrm{sech}\zeta_k e^{i\phi_k},$$
$$\zeta_k(x,t)=2\nu_k[x-\xi_k(t)], \tag{4}$$
$$\xi_k(t)=2\mu_k t+\xi_{k,0},\quad \phi_k(x,t)=\frac{\mu_k}{\nu_k}\zeta_k+\delta_k(t),$$
$$\delta_k(t)=w_k t+\delta_{k,0}, \tag{5}$$

where amplitude ν_k, velocity μ_k, c.m. position ξ_k and phase δ_k for $k=1,\cdots,N$ are the $4N$ soliton parameters. Because in the process of formation of the BEC N-soliton train, the number of solitons, the amplitude of an individual soliton, and the separation between neighbouring solitons produced should vary with the scattering length and the initial size on the condensate,[25−27] for the adiabatic approximation the soliton parameters must satisfy[28]

$$|\nu_k-\nu_0|\ll\nu_0, |\mu_k-\mu_0|\ll\mu_0,$$
$$|\nu_k-\nu_0||\xi_{k+1,0}-\xi_{k,0}|\ll 1, \tag{6}$$

where $\nu_0=\frac{1}{N}\sum_{k=1}^{N}\nu_k$ and $\mu_0=\frac{1}{N}\sum_{k=1}^{N}\mu_k$ are the average amplitude and velocity, respectively. The adiabatic approximation uses as a small parameter $\varepsilon_0\ll 1$ the soliton overlap which falls off exponentially with the distance between the solitons. The small parameter ε_0 can be related to the initial distance $r_0=|\xi_2-\xi_1|_{t=0}$ between the two solitons. Assuming $\nu_{1,2}\cong\nu_0$, we find $\varepsilon_0=\int_{-\infty}^{\infty}dx|u_1^{1s}(x,0)u_2^{1s}(x,0)|\approx 8\nu_0 r_0 e^{-2\nu_0 r_0}$. In particular, it means that $\varepsilon_0\cong 0.01$ for $r_0\cong 8$ and $\nu_0=1/2$.

In the adiabatic approximation, the dynamics of the soliton parameters can be determined by the system[14,15]

$$\frac{d(\mu_k+i\nu_k)}{dt}=-4\nu_0(e^{Q_{k+1}-Q_k}-e^{Q_k-Q_{k-1}})$$
$$+\wedge_k+i\Pi_k, \tag{7}$$
$$\frac{d\xi_k}{dt}=2\mu_k+\Theta_k,\quad \frac{d\delta_k}{dt}=2(\mu_k^2+\nu_k^2)+\Sigma_k, \tag{8}$$

where $Q_k(t)=2i\lambda_0\xi_k(t)+2k\ln(2\nu_0)+i[k\pi-\delta_k(t)-\delta_0]$, $\sum_k=2\mu_k\Theta_k+Y_k$, $\lambda_0=\mu_0+i\nu_0$, $\xi_0=\frac{1}{N}\sum_{k=1}^{N}\xi_k$, and $\delta_0=\frac{1}{N}\sum_{k=1}^{N}\delta_k$. The right-hand sides of Eqs. (7) and (8) are determined by $R[u]$ though

$$\Pi_k=\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\zeta_k}{\cosh\zeta_k}\mathrm{Re}(\mathrm{R[u]e}^{-\mathrm{i}\phi_k}),$$
$$\wedge_k=\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\zeta_k\sinh\zeta_k}{\cosh^2\zeta_k}\mathrm{Im}(\mathrm{R[u]e}^{-\mathrm{i}\phi_k}), \tag{9}$$
$$\Theta_k=\frac{1}{4\nu_k^2}\int_{-\infty}^{\infty}\frac{\zeta_k d\zeta_k}{\cosh\zeta_k}\mathrm{Re}(\mathrm{R[u]e}^{-\mathrm{i}\phi_k}),$$
$$Y_k=\frac{1}{2\nu_k}\int_{-\infty}^{\infty}\frac{d\zeta_k(1-\zeta_k\tanh\zeta_k)}{\cosh\zeta_k}\mathrm{Im}(\mathrm{R[u]e}^{-\mathrm{i}\phi_k}). \tag{10}$$

We assume that initially the solitons are ordered in such a way that $\xi_{k+1}-\xi_k\cong r_0$. One can check $\Pi_k\cong\bigwedge_k\cong\exp(-2\nu_0|k-j|r_0)$. Therefore, the interaction terms between the kth and $(k\pm1)$st solitons will be of the order of $e^{-2\nu_0 r_0}$; the interactions between the kth and $(k\pm2)$nd soliton will of the order of $e^{-4\nu_0 r_0}\ll e^{-2\nu_0 r_0}$.

From the above analysis, for the external potential $V(x)(2)$ we obtain the results

$$\Pi_k=0,\Theta_k=0, \tag{11}$$
$$\Lambda_k=-V_0\xi_k+\frac{\pi V_1\Omega^2}{8\nu_k}\frac{1}{\sinh Z_k}\sin[\Omega(\xi_k-\bar{x})]-\frac{V_2}{2}, \tag{12}$$
$$Y_k=V_0\Big(\frac{\pi^2}{48\nu_k^2}-\xi_k^2\Big)-\frac{\pi^2V_1\Omega^2}{16\nu_k^2}\frac{\cosh Z_k}{\sinh^2 Z_k}$$
$$\cdot\cos[\Omega(\xi_k-\bar{x})]-V_2\xi_k, \tag{13}$$

$\sum_k=Y_k$, and $Z_k=\Omega\pi/(4\nu_k)$. As a result, the corresponding PCTC model to describe the dynamics of a BEC N-soliton train trapped in an applied combined external potential consisting of both a harmonic and tilted periodic component in terms of soliton parameters has the form

$$\frac{d\mu_k}{dt}=16\nu_0^3[e^{-2\nu_0(\xi_{k+1}-\xi_k)}\cos\Phi_k$$
$$-e^{-2\nu_0(\xi_k-\xi_{k-1})}\cos\Phi_{k-1}]+\Lambda_k, \tag{14}$$
$$\frac{d\nu_k}{dt}=16\nu_0^3[e^{-2\nu_0(\xi_{k+1}-\xi_k)}\sin\Phi_k$$
$$-e^{-2\nu_0(\xi_k-\xi_{k-1})}\sin\Phi_{k-1}], \tag{15}$$
$$\frac{d\xi_k}{dt}=2\mu_k, \tag{16}$$
$$\frac{d\delta_k}{dt}=2(\mu_k^2+\nu_k^2)+Y_k, \tag{17}$$
$$\Phi_k=2\mu_0(\xi_{k+1}-\xi_k)+\delta_k-\delta_{k+1}. \tag{18}$$

From Eqs. (15) and (18) we find that $\frac{d\nu_0}{dt}=0$.

To verify the adequacy of the PCTC model for the description of the N-soliton train dynamics in external potentials, we perform a comparison of predictions of the corresponding PCTC model and direct simulations of the underlying NLS equation (3) with Eq. (2). The perturbed NLS Eq. (3) is solved by the operator splitting procedure using the fast Fourier transform. The corresponding PCTC model is solved by the Runge–Kutta scheme with the adaptive stepsize control. Next, we use mainly the set of parameters most widely used in numeric simulations, i.e.,

$\nu_k(0) = \dfrac{1}{2}$, $\mu_k(0) = 0$, $\xi_{k+1}(0) - \xi_k(0) = r_0$, with two different choices for the phases

$$\delta_k(0) = k\pi, \tag{19}$$

$$\delta_k(0) = 0. \tag{20}$$

Here we are interested in the following three cases.

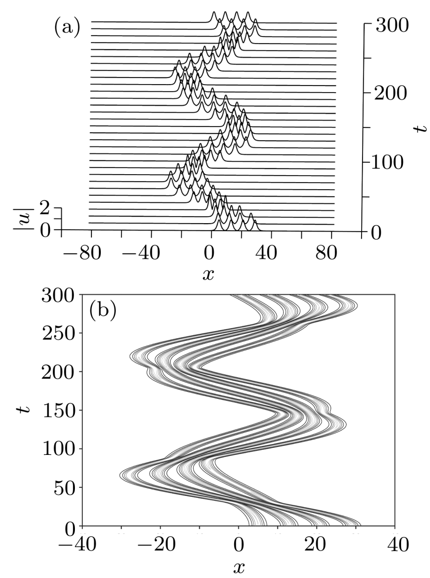

Fig. 1. (a) Evolution of the PCTC model (14)–(17) showing harmonic oscillations of a four-soliton train initially shifted relative to the minimum of the parabolic potential. The initial conditions are $(\nu_j, \mu_j, \xi_j, \delta_j) = (1/2, 0, (8j-3), (j-1)\pi)$, where $j = 1, 2, 3, 4$. (b) Contour plot of the governing equations (3) with Eq. (2). The initial input soliton train is $u(x, t = 0) = \mathrm{sech}(\mathrm{x}-5)\cdot\exp(\mathrm{i}\cdot 0\cdot\pi) + \mathrm{sech}(\mathrm{x}-13)\cdot\exp(\mathrm{i}\cdot 1\cdot\pi) + \mathrm{sech}(\mathrm{x}-21)\cdot\exp(\mathrm{i}\cdot 2\cdot\pi) + \mathrm{sech}(\mathrm{x}-29)\cdot\exp(\mathrm{i}\cdot 3\cdot\pi)$. The parameters are the train as the same as in Eq. (19) with $r_0 = 8$, $V_0 = 0.001$ and $V_1 = V_2 = 0$.

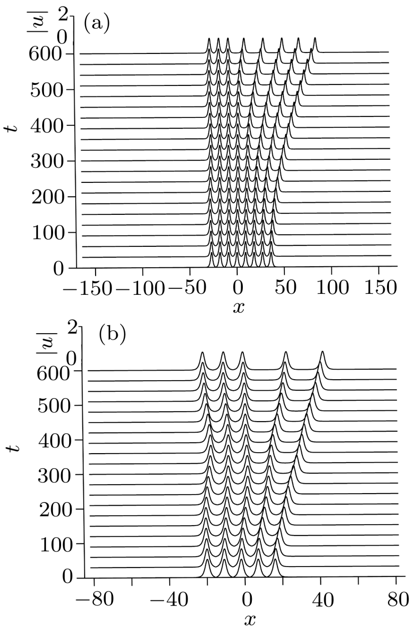

Fig. 2. The evolution from an N-soliton train with parameters as in Eq. (19) with $r_0 = 9$, $V_0 = 0$, $V_1 = -0.0005$, and $\Omega = 2\pi/9$: (a) five solitons from an eight-soliton train at $\bar{x} = 0$ and $V_2 = -0.0002$, with the initial conditions $(\nu_j, \mu_j, \xi_j, \delta_j) = (1/2, 0, 9j, (j+3)\pi)$, for $j = -3, -2, -1, 0, 1, 2, 3, 4$, (b) two solitons from a five-soliton train at $\bar{x} = -2.0$ and $V_2 = 0$, with the initial conditions $(\nu_j, \mu_j, \xi_j, \delta_j) = (1/2, 0, (9j-2), (j+2)\pi)$, for $j = -2, -1, 0, 1, 2$.

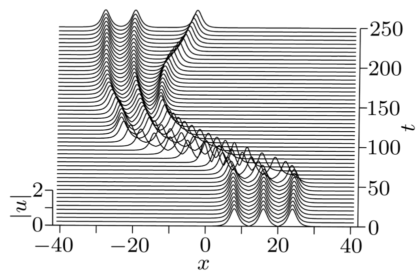

Fig. 3. Evolution of the PCTC model (14)–(17) showing the controlled placement of a three-soliton train from one lattice site to another and the controlled extraction of solitons from the train by adjusting the strength of the tilted potential with initial parameters as in Eq. (19) with $r_0 = 8$. The parameter values are: in the time range of $0 \le t < 50.0$, $V_0 = 0$, $V_1 = -0.01$, $\Omega = \pi/4$, $\bar{x} = 0$ and $V_2 = 0$; within the time range of $50 \le t < 120.2$, $V_0 = 0.001$ and $V_1 = V_2 = 0$, at time $t = 50.0$, the tilted periodic is turned off; in the time range of $t \ge 120.2$, $V_0 = 0$, $V_1 = -0.01$, $\Omega = \pi/4$, and $\bar{x} = 0$, the tilted periodic potential is turned back on $t = 70.2$ time units later thus trapping the train near the point on the other side of the origin. Depending on the tilt, a different number of solitons can be pulled out of the train: one soliton from the train at $V_2 = -0.00043$. The initial conditions are $(\nu_j, \mu_j, \xi_j, \delta_j) = (1/2, 0, 8j, (j-1)\pi)$, where $j = 1, 2, 3$.

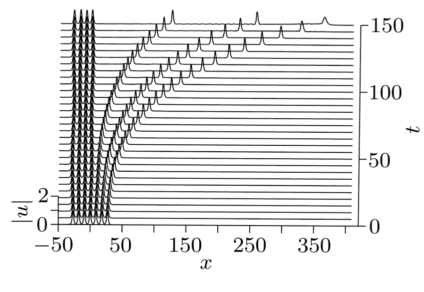

Fig. 4. Evolution of the PCTC model (14)–(17) showing the controlled extraction of solitons from a seven-soliton train by adjusting the strength of the tilted potential with parameters as in Eq. (20) with $r_0 = 9$, $V_0 = -0.00022$, $V_1 = -0.02$, $\Omega = 2\pi/9$, and $\bar{x} = 0$. Depending on the tilt, a different number of solitons can be pulled out of the train: three solitons from a seven-soliton train at $V_2 = -0.005$. The initial conditions are $(\nu_j, \mu_j, \xi_j, \delta_j) = (1/2, 0, 9j, 0)$, where $j = -3, -2, -1, 0, 1, 2, 3$.

Firstly, we only consider the BEC N-soliton train dynamics in an attractive parabolic potential in the real experiment.[23] Figure 1 shows good agreement between the PCTC model (14)–(17) and the numerical solution of the perturbed NLS equation (3) with Eq. (2). It also demonstrates effects of the attractive parabolic potential on the motion of the N-soliton train: the train oscillates around the minimum of the potential as a whole if its centre of mass is shifted. It can be seen that expulsive interactions between neighbouring solitons with parameters (19) can be balanced by attractive force on solitons, so that they remain bounded by the potential. It is worth noting that the periods of these motions can be easily changed by adjustment of the strength of the attractive parabolic potential V_0. Our results are in remarkable agreement with the predictions in Ref. [13].

Secondly, we only consider the BEC N-soliton train dynamics in a tilted periodic potential in a one-dimensional accelerated optical lattice.[18,29] Fig. 2 demonstrates that an N-soliton train confined in the potential can be flexibly manipulated by adjustment of the strength of the tilted potential and the offset parameter $\bar{x}$, respectively. It shows the extraction of a different number of solitons from the N-soliton train by increasing the strength of the tilted potential V_2 and the offset parameter $\bar{x}$, while the others remain bounded. Here the evolution of the BEC N-soliton train dynamics in a tilted periodic potential is more complicated since the potential is no longer asymmetric so that an asymmetry in the dynamics arises. The PCTC model provides an adequate description of the dynamics of an N-soliton train in a tilted periodic potential as well (the numerical solution of the perturbed NLS equation (3) with Eq. (2), which is not shown in Fig. 2, in agreement with the PCTC model is observed).

Finally, we consider two cases for describing the dynamics of the BEC N-soliton train in the parabolic[23] and tilted periodic potentials.[18,29]

Case 1. Previous section establishes that the PCTC model (14)–(17) provides a useful, accurate, and greatly simplified description of the governing BEC N-soliton train dynamics given by Eq. (3) with Eq. (2). Further, the computational time associated with Eqs. (14)–(17) is orders of magnitude faster than simulations of Eq. (3) with Eq. (2). The resulting theoretical insight can be used to provide more complex information on controlling the BEC N-soliton train dynamics. In the example illustrated in Fig. 3, the theoretical finding provides both a guide to prescribing the strengths of the tilted periodic and harmonic potentials, and an estimate for the oscillation period in the harmonic potential in the absence of the tilted periodic potential. The two together can be used to calculate a specific tilted periodic potential, which can trap the N-soliton train in the right of the origin. We can turn off the tilted periodic potential, calculate the half period of oscillation required to move the N-soliton train to the left of the origin, and then turn the tilted periodic potential back on, thus trapping the N-soliton train two stable positions away and extracting of a soliton from the soliton train.

Case 2. Fig. 4 shows that an N-soliton train placed in the repulsive parabolic, and one-dimensional accelerated optical lattice can be flexibly manipulated by adjustment of the strength of the tilted periodic potential. It demonstrates the extraction of a different number of solitons from the soliton train by increasing the strength of the tilted periodic potential (V_1 and V_2), while the others remain bounded. It can be seen that attractive interactions at zero phase difference between neighbouring solitons with parameters (20) can be balanced by expulsive force on solitons. It is noteworthy to stress that this phenomenon is in good agreement with the numerical solution of the perturbed NLS equation (3) with Eq. (2).

In conclusion, we have considered the dynamics of the BEC N-soliton train in the presence of the parabolic and tilted periodic potentials, and we have shown a good agreement between the analytical estimates based on the PCTC model and numerical simulations of the governing NLS equation. Moreover, these results provide insight into controlling and manipulating the BEC N-soliton train for macroscopic quantum applications.[1−12]

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