Solution of Gauss–Codazzi Equation with Applications in the Tzitzeica Equation *

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The surface in R^3 associated with the Tzitzeica equation is considered. By curvature coordinate transformation and surface imbedding, the Gauss–Codazzi equation is presented. Resorting to the solutions of the Gauss–Codazzi equation, the solution of the Tzitzeica equation is obtained under a restrictive condition.

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Many nonlinear integrable equations in the theory of solitons have a geometrical interpretation. The sine–Gordon equation, for example, describes the surfaces of constant negative Gaussian curvature in R^3. One of the major developments in the geometry of soliton theory was taken by the pioneering work of Sym,[1,2] who introduced the theory of soliton surfaces with the tool known as the Sym–Tafel formula.[3] A more general formula for surfaces was proposed in Ref. [4] by the immersion of a two-dimensional surface into a three-dimensional Euclidean space. The problem of the immersion is related to the problem of studying surfaces in Lie groups and surfaces in Lie algebras. In this Letter, we consider the Tzitzeica equation[4−6]

$$w_{xy} = e^{w} - e^{-2w}, \tag{1}$$

which is the nearest relative of the sine–Gordon equation. The discussion of Eq. (1) in a solitonic context were taken in Ref. [7–10]. Lie algebra[11−14] and geometry[15] have been used to discuss the relation between the two-dimensional Toda equation and the Tzitzeica equation; the latter is a special reduction of the former. The Tzitzeica equation and its linear representation are derived in their original geometric context and can be found in Ref. [15] and references therein.

In this Letter, based on the theory of n-orthogonal coordinate systems,[16−18] we give a curvature coordinate transformation and obtain the solution to the Tzitzica equation by using the imposition of the condition on the components of the quadratic forms and the results about the Gauss–Codazzi equations of the components of the quadratic forms.

We consider the Tzitzeica equation (1), which is the compatible condition of the linear equations[9,19]

$$\psi_x = \begin{pmatrix} 0 & 0 & \lambda e^{w} \\ e^{-w} & w_x & 0 \\ 0 & 1 & 0 \end{pmatrix} \psi,$$

$$\psi_y = \begin{pmatrix} w_y & e^{-w} & 0 \\ 0 & 0 & e^{w} \\ \lambda^{-1} & 0 & 0 \end{pmatrix} \psi. \tag{2}$$

If we set $\varphi = (\varphi_1, \varphi_2, \varphi_3)^{\mathrm{T}}$, $\psi = (\psi_1, \psi_2, \psi_3)^{\mathrm{T}}$, $\phi = (\phi_1, \phi_2, \phi_3)^{\mathrm{T}}$ to be the basic solutions to Eq. (2), then we can give the vector $\boldsymbol{r} = (\varphi_3, \psi_3, \phi_3)^{\mathrm{T}}$, which defines a surface Σ in R^3. By using Eq. (2), we can obtain

$$\boldsymbol{r}_x = \begin{pmatrix} \varphi_2 \\ \psi_2 \\ \phi_2 \end{pmatrix}, \quad \boldsymbol{r}_y = \lambda^{-1} \begin{pmatrix} \varphi_1 \\ \psi_1 \\ \phi_1 \end{pmatrix}. \tag{3}$$

At a point O on the surface Σ, the vectors $\boldsymbol{r}_x, \boldsymbol{r}_y$ span a tangent space. The unit length vector at the point is defined by $\boldsymbol{n} = \boldsymbol{r}_x \times \boldsymbol{r}_y$, then $\{\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{n}\}$ consist of a coordinates system in the space R^3. Accordingly, the vector $\boldsymbol{r}$ can be discomposed into

$$\boldsymbol{r} = r^1 \boldsymbol{r}_x + r^2 \boldsymbol{r}_y + \rho \boldsymbol{n}. \tag{4}$$

From Eqs. (2)–(4), we can obtain the Gauss equations of the surface Σ as

$$\begin{aligned} \boldsymbol{r}_{xx} &= w_x \boldsymbol{r}_x + \lambda e^{-w} \boldsymbol{r}_y, \\ \boldsymbol{r}_{xy} &= e^{w} r^1 \boldsymbol{r}_x + e^{w} r^2 \boldsymbol{r}_y + \rho e^{w} \boldsymbol{n}, \\ \boldsymbol{r}_{yy} &= \lambda^{-1} e^{-w} \boldsymbol{r}_x + w_y \boldsymbol{r}_y. \end{aligned} \tag{5}$$

If we set the first and second quadratic forms of the surface Σ to be

$$\begin{aligned} \mathrm{I} &= E dx^2 + 2F dxdy + G dy^2, \\ \mathrm{II} &= e dx^2 + 2f dxdy + g dy^2, \end{aligned}$$

then from Eq. (5) we have $e = g = 0, f = \rho e^{w}$ and the equations about the components of the first quadratic form

$$\begin{aligned} E_x &= 2 w_x E + 2\lambda e^{-w} F, \\ G_y &= 2 w_y G + 2\lambda^{-1} e^{-w} F. \end{aligned} \tag{6}$$

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Noticing the forms of Eq. (6), without loss of generality, we set $E = \frac{\alpha^2}{\beta^2}G$, here α, β are constants, then

$$G = \sigma(x,y)e^{2w}, \quad E = \frac{\alpha^2}{\beta^2}\sigma(x,y)e^{2w},$$
$$F = \frac{\alpha^2}{2}\sigma(x,y)e^{3w},$$

where $\sigma(x,y) = \exp(\lambda\beta^2 x + \lambda^{-1}\alpha^2 y)$.

One can find that $\boldsymbol{r}_x$ and $\boldsymbol{r}_y$ are the asymptotic directions of the surface Σ. In this context, it is more convenient to parametrize the surface in terms of curvature coordinates

$$x_1 = \alpha x + \beta y, \quad x_2 = \alpha x - \beta y, \tag{7}$$

then the first and second quadratic forms are diagonal

$$\begin{aligned} \mathrm{I} &= p^2 dx_1^2 + q^2 dx_2^2, \\ \mathrm{II} &= Apdx_1^2 + Bqdx_2^2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} p^2 &= \zeta\Big(\frac{1}{2\beta^2}e^{2w} + \frac{\alpha}{4\beta}e^{3w}\Big), \\ q^2 &= \zeta\Big(\frac{1}{2\beta^2}e^{2w} - \frac{\alpha}{4\beta}e^{3w}\Big), \end{aligned} \tag{9}$$

with

$$\zeta = \exp\Big\{\Big(\frac{\lambda\beta^2}{2\alpha} + \frac{\alpha^2}{2\lambda\beta}\Big)x_1 + \Big(\frac{\lambda\beta^2}{2\alpha} - \frac{\alpha^2}{2\lambda\beta}\Big)x_2\Big\}.$$

The coefficients of these two quadratic forms of Eq. (8) cannot be chosen independently. They are connected by three nonlinear partial differential equations known as the Gauss–Codazzi equation and a constrained condition

$$Ap = -Bq = \frac{\rho}{2\alpha\beta}e^{w}. \tag{10}$$

It is convenient to present here an alternative expression for the Gauss–Codazzi equation on introduction of two new functions a and b:

$$\frac{\partial p}{\partial x_2} = aq, \quad \frac{\partial q}{\partial x_1} = bp, \tag{11}$$

then

$$\frac{\partial A}{\partial x_2} = aB, \quad \frac{\partial B}{\partial x_1} = bA, \tag{12}$$
$$\frac{\partial a}{\partial x_2} + \frac{\partial b}{\partial x_1} + AB = 0. \tag{13}$$

Let us embed the surface Σ in R^3, which can be carried out by constructing in the vicinity of Σ in a special three-orthogonal coordinate system, then

$$\begin{aligned} ds^2 &= \sum_{i,j=1}^{3} g_{ij}dx_i dx_j \\ &= H_1^2dx_1^2 + H_2^2dx_2^2 + H_3^2dx_3^2, \end{aligned} \tag{14}$$

where H_i $(i = 1,2,3)$ are the Lamé coefficients,

$$H_1 = p + Ax_3, \quad H_2 = q + Bx_3, \quad H_3 = 1, \tag{15}$$

Apparently x_3 is directed along the normal vectors to Σ.

As far as R^3 is flat, the Riemann curvature tensor must be identically zero, from which we can obtain the following equation systems:

$$\frac{\partial Q_{ij}}{\partial x_k} = Q_{ij}Q_{kj}, \tag{16}$$
$$\frac{\partial Q_{ij}}{\partial x_j} + \frac{\partial Q_{ji}}{\partial x_i} + \sum_{k\neq i,j} Q_{ik}Q_{jk} = 0, \tag{17}$$

where

$$Q_{ij} = \frac{1}{H_j}\frac{\partial H_i}{x_j}, \quad i \neq j. \tag{18}$$

It is evident that the rotation coefficients Q_{ij} are

$$\begin{aligned} &Q_{31} = 0, \quad Q_{32} = 0, \\ &Q_{13} = A, \quad Q_{23} = B, \quad Q_{12} = a, \quad Q_{21} = b. \end{aligned} \tag{19}$$

Fortunately, the explicit solution to the Gauss–Codazzi equations (11)–(13) have been given by using the $\bar{\partial}$-dressing method[16–18] as

$$\begin{aligned} a &= Q_{12} = -\frac{2c_1'(x_1)c_2(x_2)}{\Delta}, \\ b &= Q_{21} = -\frac{2c_1(x_1)c_2'(x_2)}{\Delta}, \\ A &= -\frac{2c_1'(x_1)}{\Delta}, \quad B = -\frac{2c_2'(x_2)}{\Delta}, \end{aligned} \tag{20}$$

with

$$\begin{aligned} c_i(x_i) &= \frac{1}{\sqrt{2\pi}}\int f_i(\mu,\bar{\mu})e^{-\mu x_i}\mathrm{d}\mu\mathrm{d}\bar{\mu}, \\ \Delta &= 1 + c_1^2(x_1) + c_2^2(x_2), \end{aligned} \tag{21}$$

where $c_i'(x_i)$ denotes the derivative of function c_i with respect to x_i. Furthermore, the explicit expressions about p and q can be given by

$$\begin{aligned} p &= \langle \xi_1 e^{\lambda x_1}\rangle + \Omega\eta_1, \\ q &= \langle \xi_2 e^{\lambda x_2}\rangle + \Omega\eta_2, \end{aligned} \tag{22}$$

where ξ_i $(i = 1,2)$ are arbitrary functions, which satisfy $\xi_i(\lambda,\bar{\lambda}) = \bar{\xi}_i(\bar{\lambda},\lambda)$, and $\Omega = \langle h_1 e^{\lambda x_1}\xi_1\rangle + \langle h_2 e^{\lambda x_2}\xi_2\rangle$ with $\langle\cdot\rangle = \int \cdot d\lambda d\bar{\lambda}$.

In order to obtain the solution of the Tzitzeica equation (1), A, B, p, q in Eqs. (20) and (22) should satisfy the constrained condition (10), which is also the restriction imposed on the functions ξ_i and f_i in

Eq. (22). Then from Eq. (9), we can obtain the solution of the Tzitzeica equation (1),

$$w = \ln \frac{2(p^2 - q^2)}{\alpha\beta(p^2 + q^2)}, \tag{23}$$

where the coordinates transformation (7) should be used in p and q.

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