Exact Solutions for String Cosmology *

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Some more general cosmological solutions of Bianchi types II, VIII, and IX for a cloud string are presented. The physical implications of the solutions are briefly discussed. Our solutions include some of the results previously given in the literature as special cases.

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Recently the problem of finding the cosmological solutions of Einstein equation has attracted wide attention[1−9] and it is a challenging problem to determine the exact physical situation at the very early stages of the formation of our universe. Among the various topological defects occurred during the phase transition and before the creation of particles in the early universe, strings have interesting cosmological consequences and has been studied in more detail.[3]

The world sheets of strings are two-dimensional time-like surfaces.[4] As one of the approaches to the problem of large scale structures of the universe, one can try to explain the present-day configurations of the universe by the large-scale network of strings in the early universe. Moreover, the galaxy formation and the double quasar problem can be explained by density fluctuations of vacuum strings.

As the stress-energy of a string can be coupled to the gravitational field, it may be interesting to study the gravitational effects that arise from strings. In fact, the general relativistic treatment of strings was initiated by Letelier.[1] This model has been used as a source for Bianchi type I and Kantowski–Sachs cosmologies. Afterwards, Krori *et al.*[2] have discussed the solutions of Bianchi types II, VI. VIII and IX for a cloud string, Vilenkin[3] and Chakraborty and Chakraborty[4] have presented the exact solutions of Bianchi type III and spherically symmetric cosmology, respectively, for a cloud string.

In this Letter, we study the Letelier model in the context of Bianchi types II, VIII and IX, and more general solutions are presented. The solutions include the results previously given by Krori *et al.*[2] as special cases.

The Einstein equation for a cloud of strings is[1]

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=-(\rho u_\mu u_\nu-\lambda x_\mu x_\nu), \tag{1}$$

where we choose the units such that $c=1$, $8\pi G=1$, and ρ is the rest energy density of the cloud of strings with particles attached to them, $\rho=\rho_p+\lambda$ with ρ_p being the rest energy density of particles and λ being the tension density of the cloud of strings. As pointed out by Letelier, the energy density for the coupled system ρ and ρ_p is restricted to be positive, while the tension density λ may be positive or negative. The vector u^μ describes the four-dimensional cloud velocity and x^μ represents a direction of anisotropy, i.e., the direction of the string satisfy the standard relations,[6]

$$u^\mu u_\mu=-\chi^\mu\chi_\mu=1, \quad \text{and} \quad u^\mu x_\mu=0. \tag{2}$$

In the following, we first study the Bianchi type II case, and then move on to the case of the Bianchi types VIII and IX. The locally rotationally symmetric (LRS) metric for the spatially homogeneous Bianchi type II cosmological model is[2]

$$\mathrm{d}s^2=\mathrm{d}t^2-(S\mathrm{d}x+Sz\mathrm{d}y)^2-(R\mathrm{d}y)^2-(R\mathrm{d}z)^2, \tag{3}$$

where R and S are only the functions of t. With the help of Eqs. (2) and (3), Einstein Eqs. (1) can be written as[2]

$$2\frac{\dot R}{R}\frac{\dot S}{S}+\frac{\dot R^2}{R^2}-\frac{1}{4}\frac{S^2}{R^4}=\rho, \tag{4}$$

$$2\frac{\ddot R}{R}+\frac{\dot R^2}{R^2}-\frac{3}{4}\frac{S^2}{R^4}=\lambda, \tag{5}$$

$$\frac{\ddot S}{S}+\frac{\ddot R}{R}+\frac{\dot S}{S}\frac{\dot R}{R}+\frac{1}{4}\frac{S^2}{R^4}=0. \tag{6}$$

One notes that Eqs. (4)–(6) connect four unknown variables (R, S, λ and ρ). Thus, to be able to solve these equations we need one more relation connecting these variables. In order to obtain more general solutions, we adopt the assumption that is a relation between variables R and S, i.e., specifically

$$R=S^m, \tag{7}$$

where m is constant. Under the assumption Eq. (7), Eq. (6) can be written as

$$\ddot S+\frac{m^2}{m+1}\frac{\dot S^2}{S}+\frac{1}{4(m+1)}S^{3-4m}=0. \tag{8}$$

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To solve Eq. (8), we denote $\dot{S} = \eta$, then $\ddot{S} = \eta \dfrac{d\eta}{dS}$ and Eq. (8) can be reduced to the first-order differential equation as follows,

$$\frac{d\eta}{dS} + \frac{m^2}{m+1}\frac{\eta}{S} + \frac{1}{4(m+1)} S^{3-4m}\eta^{-1} = 0. \tag{9}$$

This is the Bernouilli equation and its solution is[10]

$$\eta = S^{-\frac{m^2}{m+1}}\Big[\frac{1}{4(m^2-2)} S^{\frac{4-2m^2}{m+1}} + C_1\Big]^{1/2}, \tag{10}$$

where C_1 is a constant of integration. Then the solution of Eq. (8) can now be expressed as

$$t - t_0 = \int \Big[\frac{1}{4(m^2-2)} S^{\frac{4-2m^2}{m+1}} + C_1\Big]^{-1/2} S^{\frac{m^2}{m+1}} dS. \tag{11}$$

Putting the constant $C_1 = 0$, from Eqs. (11) and (7) we have the functions S and R in term of the variable t,

$$S = \Big(\frac{2m-1}{2\sqrt{m^2-2}}\Big)^{\frac{1}{2m-1}} (t-t_0)^{\frac{1}{2m-1}}, \tag{12}$$

$$R = \Big(\frac{2m-1}{2\sqrt{m^2-2}}\Big)^{\frac{m}{2m-1}} (t-t_0)^{\frac{m}{2m-1}}, \tag{13}$$

Substituting Eqs. (12) and (13) into Eqs. (4) and (5), we obtain

$$\rho = \frac{2(m+1)}{(2m-1)^2}(t-t_0)^{-2}, \tag{14}$$

$$\lambda = \frac{6+2m-4m^2}{(2m-1)^2}(t-t_0)^{-2}, \tag{15}$$

$$\rho_p = \frac{4(m^2-1)}{(2m-1)^2}(t-t_0)^{-2}. \tag{16}$$

Clearly, there is a simple equation of state, i.e.,

$$\rho = k\lambda, \tag{17}$$

where $k = \dfrac{1}{3-2m}$ is constant.

Now for each given value of m we have a different model of cosmology. From Eq. (12) one sees that for S being an increasing function of t, a restriction has to be imposed on the constant m, namely $m > \sqrt{2}$, in which the energy density ρ and ρ_p are always positive, but the tension density λ is not always positive. In fact, λ is positive when $\sqrt{2} < m < 3/2$, and λ is negative when $m > 3/2$. In particular, when $m = 9/4$, we obtain the results previously given by Krori *et al.*[2] When $m = 3/2$, then $\lambda = 0$, our solutions reduce to a model in which the matter content is due to a cloud of particles only.

According to Refs. [1, 2], when $\rho_p/|\lambda| > 1$, in the process of evolution, the universe is dominated by massive strings, and when $\rho_p/|\lambda| < 1$, the universe is dominated by the strings. Now from Eqs. (15) and (16), we obtain

$$\frac{\rho_p}{|\lambda|} = \frac{2(m-1)}{|3-m|} > 1. \tag{18}$$

Thus, in our model, throughout the whole process of evolution, the universe is dominated by massive strings.

Next we move on to the case of Bianchi types VIII and IX. The locally rotationally symmetric metric for Bianchi type VIII ($\delta = -1$) and Bianchi type IX ($\delta = +1$) is[2]

$$ds^2 = dt^2 - (Sdx - Shdz)^2 - (Rdy)^2 - (Rfdz)^2, \tag{19}$$

where R and S are the functions of t only, and f, h and δ are as follows

$$f(y) = \begin{bmatrix} \sin y \\ \sinh y \end{bmatrix}, \quad h(y) = \begin{bmatrix} \cos y \\ -\cosh y \end{bmatrix}, \quad \text{for } \delta = \begin{bmatrix} +1 \\ -1 \end{bmatrix}. \tag{20}$$

Based on Eqs. (19) and (20), Einstein Eq (1) can be written as

$$2\frac{\dot{S}}{S}\frac{\dot{R}}{R} + \frac{1}{R^2}(\dot{R}^2 + \delta) - \frac{1}{4}\frac{S^2}{R^4} = \rho, \tag{21}$$

$$2\frac{\ddot{R}}{R} + \frac{1}{R^2}(\dot{R}^2 + \delta) - \frac{3}{4}\frac{S^2}{R^4} = \lambda, \tag{22}$$

$$\frac{\ddot{S}}{S} + \frac{\ddot{R}}{R} + \frac{\dot{S}}{S}\frac{\dot{R}}{R} + \frac{1}{4}\frac{S^2}{R^4} = 0. \tag{23}$$

It is obvious that Eq. (23) is the same as Eq. (6). Thus in the same way as has been performed for the Bianchi type II case, still using the assumption Eq. (7), we obtain the solutions to Eqs. (21)–(23), i.e.,

$$S = \Big(\frac{2m-1}{2\sqrt{m^2-2}}\Big)^{\frac{1}{2m-1}} (t-t_0)^{\frac{1}{2m-1}}, \tag{24}$$

$$R = \Big(\frac{2m-1}{2\sqrt{m^2-2}}\Big)^{\frac{m}{2m-1}} (t-t_0)^{\frac{m}{2m-1}}, \tag{25}$$

$$\rho = \frac{2(m+1)}{(2m-1)^2}(t-t_0)^{-2} + \delta\Big(\frac{2\sqrt{m^2-2}}{2m-1}\Big)^{\frac{m}{2m-1}}(t-t_0)^{-\frac{2m}{2m-1}}, \tag{26}$$

$$\lambda = \frac{6+2m-4m^2}{(2m-1)^2}(t-t_0)^{-2} + \delta\Big(\frac{2\sqrt{m^2-2}}{2m-1}\Big)^{\frac{2m}{2m-1}}(t-t_0)^{-\frac{2m}{2m-1}}, \tag{27}$$

$$\rho_p = \frac{4m^2-4}{(2m-1)^2}(t-t_0)^{-2}. \tag{28}$$

Such as in the case of Bianchi type II, we have different models of cosmology for each given value of

m. Also, as required from Eq. (24) we have the restriction on the constant m, i.e., $m > \sqrt{2}$. Especially when $m = 9/4$, we have the results previously given by Krori *et al.* Afterwards from Eqs. (27) and (28), we can easily obtain at $t \to t_0$,

$$\frac{\rho_p}{|\lambda|} \to \frac{2(m-1)}{|3-2m|} > 1, \tag{29}$$

and at $t \to \infty$, we have

$$\frac{\rho_p}{|\lambda|} \to \frac{4(m^2-1)(2m-1)^{\frac{2-2m}{2m-1}}}{(2\sqrt{m^2-2})^{\frac{2m}{2m-1}}} \cdot (t-t_0)^{-\frac{2m-2}{2m-1}} \to 0. \tag{30}$$

As shown by Eq. (29), in the early epoch of evolution the universe is dominated by massive strings as the matter content. However, in the later phase, according to Eq. (30), the strings dominate over the particles.

In summary, some more general cosmological solutions have been obtained for Bianchi types II, VIII, and IX, with a cloud string as the matter content. Our general solutions include some of the results previously given in the literature as special cases.

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