Peakons and Pseudo-Peakons of Higher Order b-Family Equations

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This paper explores the rich structure of peakon and pseudo-peakon solutions for a class of higher-order b-family equations, referred to as the J-th b-family (J-bF) equations. We propose several conjectures concerning the weak solutions of these equations, including a b-independent pseudo-peakon solution, a b-independent peakon solution, and a b-dependent peakon solution. These conjectures are analytically verified for $J \leq 14$ and/or $J \leq 9$ using the symbolic computation system MAPLE, which includes a built-in definition of the higher-order derivatives of the sign function. The b-independent pseudo-peakon solution is a 3rd-order pseudo-peakon for general arbitrary constants, with higher-order pseudo-peakons derived under specific parameter constraints. Additionally, we identify both b-independent and b-dependent peakon solutions, highlighting their distinct properties and the nuanced relationship between the parameters b and J. The existence of these solutions underscores the rich dynamical structure of the J-bF equations and generalizes previous results for lower-order equations. Future research directions include higher-order generalizations, rigorous proofs of the conjectures, interactions between different types of peakons and pseudo-peakons, stability analysis, and potential physical applications. These advancements significantly contribute to the understanding of peakon systems and their broader implications in mathematics and physics.

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1. Introduction. The b-family of equations represents a significant class of nonlinear partial differential equations that have garnered substantial attention in the field of mathematical physics. These equations are known for their rich structure and the presence of peakon solutions, which are solitary waves with a peak at their crest. The general form of the b-family equation is given by

$$m_t + v m_x + b v_x m = 0, \quad m = (1 - \partial_x^2) v, \tag{1}$$

where b is a parameter that influences the nonlinearity and dispersion properties of the equation. The study of the b-family equations has provided significant insights into nonlinear wave behavior, particularly through the discovery of peakon solutions (weak solutions) with discontinuous first derivatives at their peaks. These solutions have been widely studied due to their unique properties and physical applications.

Camassa and Holm [1] introduced the Camassa–Holm (CH) equation, a special case of the b-family with $b = 2$, which admits peakon solutions. This foundational work spurred further research into the integrability and stability of peakons. Parker recovered the peakon solutions from the analytic solitary wave by using a limiting procedure. [2] Degasperis and Procesi [3] derived another member of the b-family for $b = 3$, the Degasperis–Procesi (DP) equation, which also supports peakons and exhibits a bi-Hamiltonian structure, emphasizing its integrability.

The concept of pseudo-peakons, smooth approximations of peakons, has further enriched the field. Research by Lenells [4] and others has explored their properties and stability, bridging the gap between smooth solitons and peakons. Constantin and Strauss [5] investigated peakon stability, while Ivanov [6] advanced the understanding of higher-order b-family equations. Holm and Staley [7] analyzed wave structures and nonlinear balances in evolutionary partial differential equations, contributing to the dynamics of peakon systems. Johnson [8] connected the Camassa–Holm and Korteweg-de Vries equations, enhancing their relevance in water wave modeling. Olver–Rosenau [9] and Chen–Liu–Zhang [10] generalized the Camassa–Holm equation to two components, expanding solution scopes and integrability insights.

Recent advancements have extended the b-family and related models to higher-order formulations. [11–13] Henry, Ivanov, and Sakellaris [11] extended the CH and DP equations, offering a theoretical framework for oceanic internal wave-current interactions. Zhu–Zhu [14] investigated single pseudo and multi-pseudo peakons of the b-family fifth-order CH equation via analytical calculations and numerical simulations, and presented some interesting phenomena of multi-pseudo peakon that do not appear in the classical CH interactions. Gorka, Pons, and Reyes [15] explored higher-order CH equations using loop group geometry, while Liu and Qiao [16] introduced pseudo-peakons and multi-peakons in a fifth-order model. Qiao and

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Reyes[17] further advanced the field with fifth-order extensions, broadening the understanding of nonlinear wave phenomena.

In this paper, we investigate the peakon and pseudo-peakon structures for a class of higher-order b-family equations, defined as

$$m_t + vm_x + bv_xm = 0, \quad m = (1 - \partial_x^2)^J v, \tag{2}$$

where J is an arbitrary positive integer. For notational convenience, we refer to Eq. (2) as the J-th b-family (J-bF) equation. Specifically, when $b = 2$, it is termed the J-th CH equation, and for $b = 3$, it is referred to as the J-th DP equation. The loop group geometric aspects of the J-CH equation have been previously explored in Ref. [15] for more general high order peakon systems with $m = \sum_{i=0}^{n} c_i\partial_x^{2i}v$.

In Section 2 of this paper, we propose several conjectures concerning the weak solutions of the J-bF system, Eq. (2). Sections 3 through 6 are dedicated to verifying the validity of these conjectures and the conclusions presented in Section 2 for the cases of $J = 2$, 3, 4, and 5, respectively. The final section provides a brief summary and additional discussion.

2. Conjectures. Through extensive and intricate calculations on the weak solutions of the J-bF system Eq. (2) for various values of J, we propose the following conjectures:

Conjecture 1. *Pseudo-Peakon Conjecture.* The J-bF Eq. (2) admits a b-independent pseudo-peakon solution,

$$\begin{aligned} &v = c\Big(1 + |\xi| + \sum_{i=1}^{J-2} a_i|\xi|^{i+1}\Big)\mathrm{e}^{-|\xi|}, \\ &\forall J \geq 3, \quad \xi \equiv x - ct, \end{aligned} \tag{3}$$

where $a_1, a_2, \ldots, a_{J-2}$ are arbitrary constants.

For $J = 2$, the pseudo-peakon solution can also be cast into the conjecture without the summation term. The conjecture has been verified for the cases of $J \leq 14$ by means of the symbolic computation system MAPLE, which includes a built-in definition of the higher-order derivatives of the sign function, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{sign}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{sign}(x)$. While a clear path to a general proof remains elusive at this stage, we are hopeful that a proof will be established by researchers in the near future.

The simplest scenario, corresponding to $J = 2$ in Eq. (3), has been previously established by several researchers. However, the generalized solution (3) for any $J > 2$ is introduced for the first time in this paper. This pseudo-peakon solution exhibits a rich and intricate structure. In the following sections, we explore specific instances for various values of J. Prior to delving into particular cases, we provide a formal definition of the nth-order pseudo-peakon.

Definition. A solution is termed the nth-order pseudo-peakon if its ith-order x-derivatives are continuous for all $i \leq n - 1$, while the nth-order x-derivative is discontinuous.

By examining the smoothness of solution (3), we can identify several properties of the pseudo-peakon of Eq. (3):

(i) The solution (3) represents a 3rd-order pseudo-peakon for general values of a_i ($i = 1, 2, \ldots, J - 2$), except in the following special cases.

(ii) If the constants a_1 and a_2 satisfy the constraint condition ($v_{xi} = v_{x^i}$), $v_{x3}|_{\xi\to0^+} = v_{x3}|_{\xi\to0^-}$, i.e.,

$$1 - 3(a_1 - a_2) = 0, \tag{4}$$

the solution (3) becomes a 5th-order pseudo-peakon without requiring additional constraints on the remaining parameters.

(iii) A 7th-order pseudo-peakon can be derived from (3) by imposing the parameter constraints (4) and

$$1 - 5a_1 + 15a_2 - 30a_3 + 30a_4 = 0, \tag{5}$$

which is obtained from $v_{x5}|_{\xi\to0^+} = v_{x5}|_{\xi\to0^-}$.

(iv) Under the parameter constraints (4) and (5), as well as

$$\begin{aligned} &1 - 7a_1 + 35a_2 - 140a_3 + 420a_4 \\ &- 840a_5 + 840a_6 = 0, \end{aligned} \tag{6}$$

which is equivalent to $v_{x7}|_{\xi\to0^+} = v_{x7}|_{\xi\to0^-}$, a 9th-order pseudo-peakon can be obtained from (3).

Higher-order pseudo-peakons can be derived from (3) by introducing additional parameter constraints which can be derived from

$$v_{x^{2j+1}}|_{\xi\to0^+} = v_{x^{2j+1}}|_{\xi\to0^-}. \tag{7}$$

For odd $J = 2n + 1$ [even $J = 2(n + 1)$], one can obtain 3rd, 5th, ..., $(4n + 1)$th [$(4n + 3)$th]-order pseudo-peakon solutions by suitable parameter conditions from (7).

In addition to the pseudo-peakon solution (3), we may identify various other types of weak solutions. Here, we present further two conjectures on peakons.

Conjecture 2. *b-Independent Peakon Conjecture.* The J-bF equation (2) admits a b-independent peakon solution of the form

$$v = c\left(1 + \sum_{i=1}^{J-1} a_i|x - ct|^i\right)\mathrm{e}^{-|x-ct|}, \quad \forall J \geq 1, \tag{8}$$

where the b-independent constants a_i satisfy the inequalities $0 < a_{J-1} < a_{J-2} < \cdots < a_2 < a_1 < 1$.

Using the symbolic computation system MAPLE which provides a built-in implementation of the higher-order derivatives of the sign function we have verify the conjecture 2 for $J \leq 9$. However, a unified formula for all J remains elusive. The first few sets of constants a_i are given by

$$\begin{aligned} &\{J = 2: \ a_1 = 2\}, \ \Big\{J = 3: \ a_1 = \frac{72}{138}, \ a_2 = \frac{13}{138}\Big\}, \\ &\Big\{J = 4: \ a_1 = \frac{495}{1136}, \ a_2 = \frac{81}{1136}, \ a_3 = \frac{1}{213}\Big\}, \\ &\Big\{J = 5: \ a_1 = \frac{16433970}{29110637}, \ a_2 = \frac{40909737}{291106370}, \\ &a_3 = \frac{8220074}{436659555}, \ a_4 = \frac{274669}{232885096}\Big\}. \end{aligned} \tag{9}$$

Conjecture 3. *b-Dependent Peakon Conjecture.* The J-bF Eq. (2) admits one real b-dependent peakon solution for odd J and two real b-dependent peakon solutions for even J, expressed in the form

$$v = c\left(\sum_{i=0}^{J-1} c_i|x - ct|^i\right)\mathrm{e}^{-|x-ct|}, \quad \forall J \geq 3, \tag{10}$$

where the b-dependent constants c_i $(i=0,1,2,\ldots,J-1)$ are determined for each J, with $c_0\neq 1$.

The validity of this conjecture has been verified for $J\leq 9$. For $J=3$, the constants c_i are given by

$$c_0=\frac{385923}{385923-67195b},\quad c_1=\frac{188717}{385923-67195b},$$
$$c_2=\frac{30038}{385923-67195b}. \tag{11}$$

For $J=4$, the constants c_i in Eq. (10) are

$$\begin{aligned}c_1&=\Big(\frac{317039585}{727147048}+\frac{9089301767}{174515291152b}\Big)c_0-\frac{9089301767}{174515291152b},\\ c_2&=\Big(\frac{78194177}{1090720572}+\frac{813796249}{32721617116b}\Big)c_0-\frac{813796249}{32721617116b},\\ c_3&=\Big(\frac{7045177}{1454294096}+\frac{3931438217}{1047091749912b}\Big)c_0\\ &\quad-\frac{3931438217}{1047091749912b},\end{aligned} \tag{12}$$

where c_0 is determined by the quadratic equation

$$\frac{19767871994017357}{55512146191680}(c_0-1)^2+\frac{17680946891729}{385501015220}bc_0(c_0-1)+b^2c_0^2=0,$$

which yields two real roots for arbitrary b.

Through extensive calculations for larger J and fixed real b, it is observed that there will exist exactly one (two) real b-dependent peakon solution(s) (10) for odd $J=2n+1$ (even $J=2n+2$), alongside $2(n-1)$ complex peakons. However, the complex peakons are not discussed in this paper.

3. Fifth Order b-Family and its Peakon and Pseudo-Peakon Solutions. For $J=2$, the J-bF system (2) becomes one of the known fifth order b-families $(v_{xi}=v_{x^i})$,

$$m_t+vm_x+bv_xm=0,\quad m=v-2v_{xx}+v_{x4}. \tag{13}$$

The single weak peakon and/or the pseudo-peakon solution of Eq. (13) can be assumed in the form

$$v=c(a_0+a_1|\xi|)\mathrm{e}^{-|\xi|},\quad \xi=x-ct, \tag{14}$$

which is hinted by vanishing m.

From Eq. (14), we have

$$\begin{aligned}v_x=&-c\,\mathrm{sgn}(\xi)(a_1|\xi|+a_0-a_1)\mathrm{e}^{-|\xi|},\\ m=&\,2\delta(\xi)\{(a_1|\xi|+a_0-2a_1)[6\delta(\xi)-4\,\mathrm{sgn}(\xi)]\\ &+5a_1|\xi|+5a_0-17a_1\}\mathrm{e}^{-|\xi|},\\ m_t=&-cm_x\\ =&-c\{8[\mathrm{sgn}(\xi)\delta'(\xi)+2\delta(\xi)^2\\ &-3\delta(\xi)\delta'(\xi)](2a_1-a_1|\xi|-a_0)\\ &-4\delta(\xi)[3\,\mathrm{sgn}(\xi)\delta(\xi)-2](a_1|\xi|+a_0-3a_1)\\ &-2(5a_1|\xi|+5a_0-22a_1)\,\mathrm{sgn}(\xi)\delta(\xi)\\ &+2(5a_1|\xi|+5a_0-17a_1)\delta(\xi)\}\mathrm{e}^{-|\xi|},\end{aligned} \tag{15}$$

where $\mathrm{sgn}(\xi)$ is the signum function of ξ, $\delta(\xi)$ is the Dirac delta function of ξ while $\delta'(\xi)$ is the derivative of $\delta(\xi)$ with respect to ξ.

Substituting Eq. (14) into Eq. (13) and requiring the result to be a zero distribution yields only two possible solutions

$$a_0=a_1=1 \tag{16}$$

and

$$a_0=1,\quad a_1=\frac{1}{2}. \tag{17}$$

The first case is just the known pseudo-peakon solution

$$v=c(1+|\xi|)\mathrm{e}^{-|\xi|},\quad \xi=x-ct, \tag{18}$$

which shares the same functional form as a solution to another fifth-order CH equation.[17] The solution (18) is classified as a 3rd pseudo-peakon but not a peakon (1st pseudo-peakon). This distinction arises because the first and second order derivatives of v are continuous, satisfying $v_x(\xi=0^+)=v_x(\xi=0^-)$ and $v_{xx}(\xi=0^+)=v_{xx}(\xi=0^-)$, while the third order derivative is discontinuous, i.e., $v_{x3}(\xi=0^+)\neq v_{x3}(\xi=0^-)$.

The second solution (17) yields a previously unknown b-independent peakon solution

$$v=\frac{1}{2}c(2+|\xi|)\mathrm{e}^{-|\xi|}, \tag{19}$$

for the 2-bF Eq. (13).

4. Seventh Order b-Family and 3rd and 5th Order Pseudo-Peakons. For $J=3$, the model (2) becomes a seventh order b-family,

$$m_t+vm_x+bv_xm=0,\quad m=v-3v_{xx}+3v_{x4}-v_{x6}. \tag{20}$$

By vanishing m of Eq. (20), one can assume that the seventh order b-family Eq. (20) admits the weak solutions in the form

$$v=c(a_0+a_1|x-ct|+a_2|x-ct|^2)\mathrm{e}^{-|x-ct|} \tag{21}$$

for possible parameter selections.

Substituting Eq. (21) into the seventh order b-family Eq. (20), one can find that requiring the result to be a zero distribution is only valid under the following conditions:

$$\begin{aligned}&(a_0-1)(a_0-3a_1+6a_2)=0,\\ &15(a_0-a_1)(a_0-3a_1+6a_2)b\\ &\ +(a_0-1)(59a_0-223a_1+643a_2)=0,\\ &2(a_0-a_1)(7a_0-34a_1+114a_2)b\\ &\ +(a_0-1)(27a_0-149a_1+646a_2)=0.\end{aligned} \tag{22}$$

From Eq. (22), it is not difficult to find three sets of solutions,

$$a_0=a_1=1, \tag{23}$$

$$a_0=1,\quad a_1=\frac{12}{23},\ a_2=\frac{13}{138}, \tag{24}$$

$$a_0=\frac{385923}{385923-67195b},\quad a_1=\frac{188717}{385923-67195b},$$
$$a_2=\frac{30038}{385923-67195b}. \tag{25}$$

By substituting solution (23) back into Eq. (21), we obtain the result of the pseudo-peakon that coincides with the conjecture (3), expressed as:

$$v=c(1+|x-ct|+a|x-ct|^2)\mathrm{e}^{-|x-ct|}, \tag{26}$$

where the arbitrary constant a_2 has been redefined as a. This formulation captures the essence of the pseudo-peakon solution, emphasizing its dependence on the arbitrary parameter a.

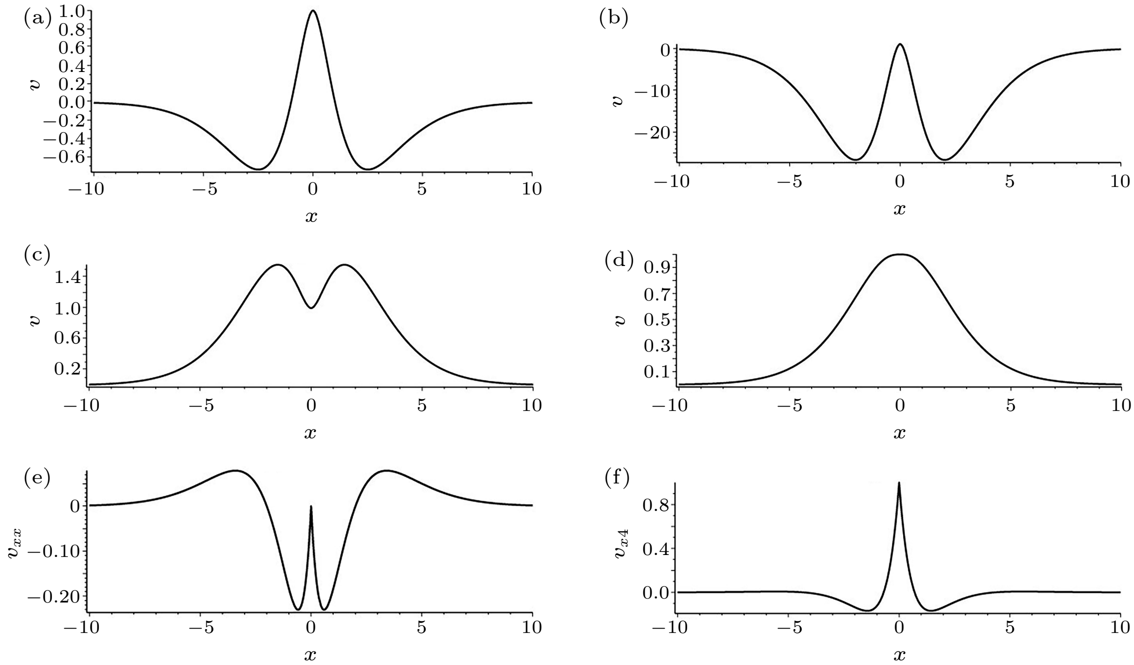

Fig. 1. (a) Pseudo-peakon structure with one smooth peak and double bottoms expressed by Eq. (21) and $a = -2$. (b) W-shaped pseudo-peakon structure given by Eq. (26) and $a = -50$. (c) M-shaped pseudo-peakon structure of Eq. (26) with $a = 2$. (d) Pseudo-peakon structure with a single smooth peak expressed by Eq. (26) and $a = \frac{1}{2}$. (e) The structure of the second-order derivative of the 3rd-order pseudo-peakon of Eq. (26) with $a = \frac{1}{2}$. (f) The structure of the fourth-order derivative of the 5th-order pseudo-peakon of Eq. (26) with $a = \frac{1}{3}$.

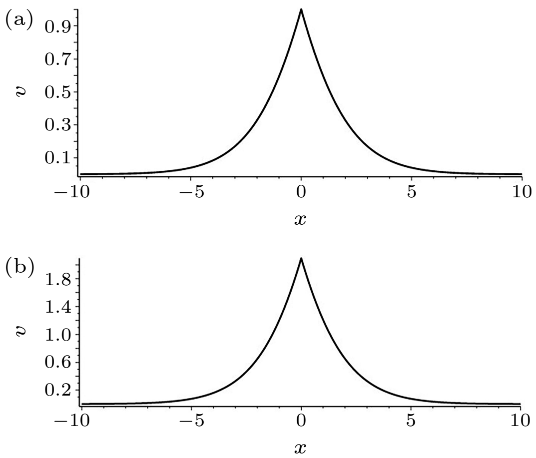

Fig. 2. The graphs of peakons. (a) The structure of the b-independent peakon expressed by Eq. (27). (b) The structure of the b-dependent peakon of Eq. (28) with $b = 3$.

The solution (26) represents a 3rd-order pseudo-peakon for a general constant $a \neq \frac{1}{3}$, while it reduces to a 5th-order pseudo-peakon for the specific constant $a = \frac{1}{3}$. Figures 1(a)–1(d) illustrate the intricate structure of the pseudo-peakon of Eq. (26) for parameter choices $a = -2$, -50, 2, and $\frac{1}{2}$, respectively, at time $t = 0$. The velocity parameter c and the time t are fixed at $c = 1$ and $t = 0$ for all figures in this paper. Figure 1(e) demonstrates the structure of the second-order partial derivative, v_{xx}, at the center of the pseudo-peakon of Eq. (26) with $a = \frac{1}{2}$. Similarly, figure 1(f) highlights the property of the fourth-order partial derivative, v_{x4}, at the center of the pseudo-peakon of Eq. (26) with $a = \frac{1}{3}$. Figures 1(e) and 1(f) also indicate the discontinuous of the third- and the fifth-order partial derivatives of the field v with respect to x.

By substituting the solution (24) into the solution ansatz (21), we derive the result of the b-independent peakon conjecture (Conjecture 2 of Section 2), which takes the following form:

$$v = \frac{1}{138} c(138 + 72|x - ct| + 13|x - ct|^2)\mathrm{e}^{-|x-ct|}. \quad (27)$$

This expression represents the peakon solution independent of the parameter b. The structure of this b-independent peakon of Eq. (27) is illustrated in Fig. 2(a) for the amplitude $c = 1$ at the time $t = 0$.

By substituting the solution (25) into the solution assumption (21), we arrive at the result of the b-dependent peakon conjecture (Conjecture 3 of Section 2), which is expressed as follows:

$$v = \frac{653c}{385923 - 67196b} \cdot (591 + 289|x - ct| + 46|x - ct|^2)\mathrm{e}^{-|x-ct|}. \quad (28)$$

This equation describes the peakon solution, highlighting its dependence on the model parameter b.

Figure 2(b) depicts the structure of this b-dependent peakon of Eq. (28) with the amplitude $\frac{385923c}{385923-67196b}$, $b = 3$, $c = 1$ at time $t = 0$. For this solution, a critical value of b exists, defined as $b_{\mathrm{cr}} = \frac{385923}{67196} \approx 5.743329$. When the parameter b transitions from $b > b_{\mathrm{cr}}$ (with $c > 0$) to $b < b_{\mathrm{cr}}$, the peakon transforms into an anti-peakon. Moreover, as b approaches the critical value b_{cr}, the amplitude of the peakon diverges to infinity.

5. Ninth Order b-Family and 3rd, 5th, and 7th Order Pseudo-Peakons. For $J = 4$, the J-bF system (2) reduces to a ninth-order b-family equation,

$$m_t + vm_x + bv_x m = 0,$$
$$m = v - 4v_{xx} + 6v_{x4} - 4v_{x6} + v_{x8}, \quad (29)$$

which admits the weak solution ansatz

$$v = c(a_0 + a_1|x-ct| + a_2|x-ct|^2 + a_3|x-ct|^3)\mathrm{e}^{-|x-ct|}. \quad (30)$$

Substituting Eq. (30) into the ninth order b-family Eq. (29), the zero distribution condition yields

$$\begin{aligned}
&60B(a_0 - 4a_1 + 12a_2 - 24a_3)b \\
&+ A(377a_0 - 1671a_1 + 6042a_2 - 16842a_3) = 0, \\
&16B(24a_0 - 107a_1 + 389a_2 - 1089a_3)b \\
&+ A(1259a_0 - 6214a_1 + 26038a_2 - 89712a_3) = 0, \\
&8B(5a_0 - 32a_1 + 176a_2 - 768a_3)b \\
&+ A(81a_0 - 553a_1 + 3320a_2 - 16872a_3) = 0, \\
&A(a_0 - 4a_1 + 12a_2 - 24a_3) = 0, \\
&A \equiv a_0 - 1,\ B \equiv a_0 - a_1.
\end{aligned} \quad (31)$$

It is evident that the simplest solution of Eq. (31), $A = B = 0$ (i.e., $a_0 = a_1 = 1$), corresponds to the pseudo-peakon solution,

$$v = c(1+|x-ct| + a_1|x-ct|^2 + a_2|x-ct|^3)\mathrm{e}^{-|x-ct|}, \quad (32)$$

which aligns with the solution presented in Conjecture 1, Eq. (3). Here, the arbitrary constants a_2 and a_3 have been redefined as a_1 and a_2, respectively.

The solution (32) corresponds to a 3rd-order pseudo-peakon for general constants $\{a_1, a_2, b, c\}$ under the condition $3(a_2 - a_1) + 1 \neq 0$. When $3(a_2 - a_1) + 1 = 0$ but $a_1 \neq \frac{2}{5}$, the solution (32) reduces to a 5th-order pseudo-peakon. Additionally, for the specific parameter values $\{a_1 = \frac{2}{5}, a_2 = \frac{1}{15}\}$, equation (32) describes a 7th-order pseudo-peakon.

Figures 3(a)–3(d) illustrate the structures of the pseudo-peakon defined by Eq. (32) for the parameter sets $\{a_1 = \frac{1}{2}, a_2 = \frac{1}{6}\}$, $\{a_1 = 1, a_2 = 1\}$, $\{a_1 = -\frac{5}{4}, a_2 = 1\}$, and $\{a_1 = 1, a_2 = -30\}$, respectively. To demonstrate the discontinuous nature of the pseudo-peakon of Eq. (32), figures 3(e) and 3(f) showcase the peak characteristics of the fourth- and sixth-order derivatives of the field v with respect to x.

The second solution of Eq. (31) corresponds to $A = 0$ and $B \neq 0$, which yields the b-independent solution (11). This solution represents the b-independent peakon solution, expressed as:

$$v = \frac{c}{3408}(3408 + 1485|x-ct| + 243|x-ct|^2 + 16|x-ct|^3)\mathrm{e}^{-|x-ct|}. \quad (33)$$

The third solution of Eq. (31) corresponds to $A \neq 0$ and $B \neq 0$, which yields the b-dependent peakon solution (12). This solutio is associated with two b-dependent

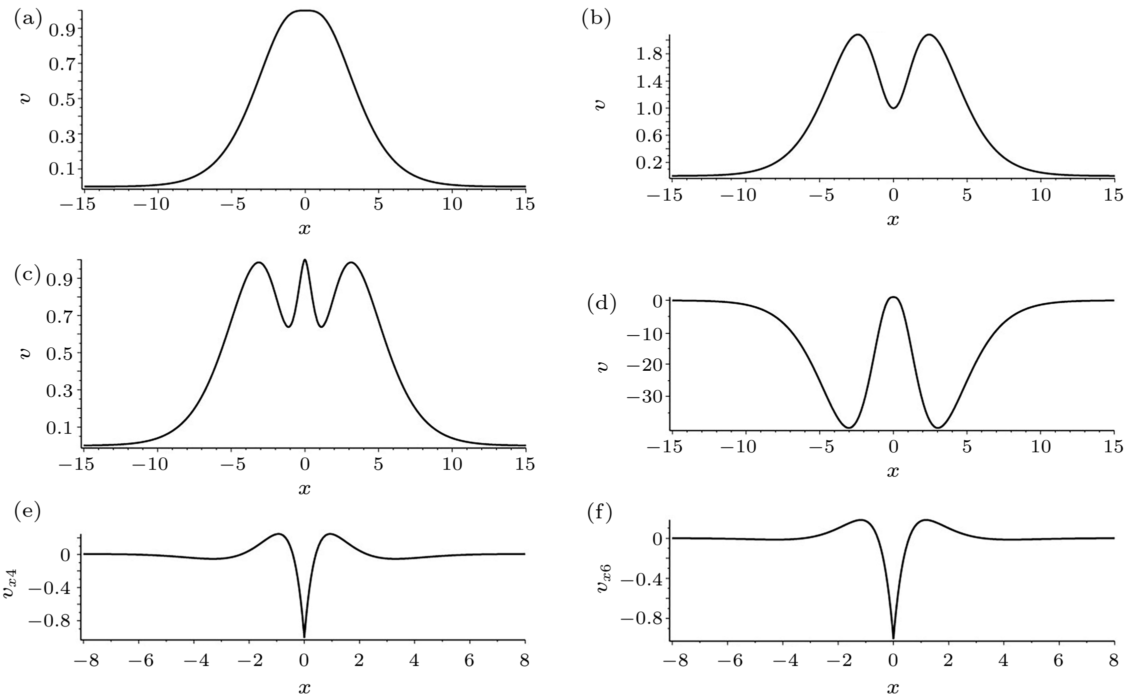

Fig. 3. (a) A quite smooth pseudo-peakon of Eq. (32) with $\{a_1 = \frac{1}{2},\ a_2 = \frac{1}{6}\}$ and $v_x|_{x=ct} = v_{xx}|_{x=ct} = v_{xxx}|_{x=ct} = 0$. (b) M-shaped pseudo-peakon of Eq. (32) with $a_1 = a_2 = 1$. (c) The structure of the pseudo-peakon with three smooth peaks expressed by (32), $a_1 = -\frac{5}{4}$ and $a_2 = 1$. (d) W-shaped pseudo-peakon Eq. (32) with $\{a_1 = 1,\ a_2 = -30\}$. (e) The structure of the fourth order derivative of v with respect to x for the 5th-order pseudo-peakon expressed by Eq. (32) with the same parameters as (a). (f) The structure of the sixth order derivative of v with respect to x for the 7th-order pseudo-peakon given by Eq. (32) with the selections $\{a_1 = \frac{2}{5},\ a_2 = \frac{1}{15}\}$.

peakon solutions, and it can be expressed in a unified form [with $\alpha \equiv 11461919(a_0-1)$] as:

$$v = \frac{c}{1047091749120b}[24a_0b(21135531|x-ct|^3 + 312776708|x-ct|^2 + 1902237510|x-ct| + 4362882288) + \alpha(343|x-ct|^3 + 2272|x-ct|^2 + 4758|x-ct|)]e^{-|x-ct|}, \tag{34}$$

where the parameter a_0 takes two distinct values,

$$a_0 = \frac{1}{27905650366969405} \cdot (23586956522630821 \pm 288\sqrt{5677557914281864097419947 4}). \tag{35}$$

The structures of the solutions (33) and (34) are qualitatively similar to those illustrated in Figs. 2(a) and 2(b).

6. Eleventh Order b-family and 3rd, 5th, 7th, and 9th Order Pseudo-Peakons. For $J=5$, the J-bF system (2) reduces to an eleventh order b-family equation,

$$m_t + vm_x + bv_xm = 0,$$
$$m = v - 5v_{xx} + 10v_{x4} - 10v_{x6} + 5v_{x8} - v_{x10}, \tag{36}$$

which admits the following weak solution ansatz:

$$v = c(a_0 + a_1|x-ct| + a_2|x-ct|^2 + a_3|x-ct|^3 + a_4|x-ct|^4)e^{-|x-ct|}. \tag{37}$$

Here, the constants a_i $(i=0,1,\ldots,4)$ are to be determined by requiring that the solution (37) satisfies the weak solution condition of Eq. (36).

Substituting Eq. (37) into the eleventh order b-family Eq. (36) and imposing the condition that the resulting equation vanishes in the distributional sense, we derive the following constraints:

$$\begin{aligned}
&90BCb + A(857a_0 - 4418a_1 + 19194a_2 - 68088a_3 + 186888a_4) = 0,\\
&2B(61a_0 - 466a_1 + 3170a_2 - 18960a_3 + 96240a_4)b + A(243a_0 - 1985a_1 + 14554a_2 - 94890a_3 + 540240a_4) = 0,\\
&5B(363a_0 - 1985a_1 + 9458a_2 - 38466a_3 + 129360a_4)b + A(5648a_0 - 33181a_1 + 172801a_2 - 786780a_3 + 3069900a_4) = 0,\\
&28B(197a_0 - 968a_1 + 3984a_2 - 13368a_3 + 34968a_4)b + A(27127a_0 - 143737a_1 + 654694a_2 - 2511654a_3 + 7899360a_4) = 0,\\
&AC = 0,\ C \equiv a_0 - 5a_1 + 20a_2 - 60a_3 + 120a_4,
\end{aligned} \tag{38}$$

where A and B are the same as in Eq. (31).

From the constraint Eq. (38), it is evident that the simplest solution $A=B=0$ leads to the result of Conjecture 1,

$$v = c(1 + |x-ct| + a_1|x-ct|^2 + a_2|x-ct|^3 + a_3|x-ct|^4)e^{-|x-ct|}, \tag{39}$$

where the renamed constants a_1, a_2, and a_3 are arbitrary.

The solution (39) represents a b-independent 3rd-order pseudo-peakon for arbitrary constants $\{a_1, a_2, a_3, b, c\}$, provided that $3(a_2-a_1)+1 \neq 0$. When $3(a_2-a_1)+1=0$ but $5(3a_3-a_1)+2 \neq 0$, the solution (39) simplifies to a 5th-order pseudo-peakon. For $\{3(a_2-a_1)+1=0, 5(3a_3-a_1)+2=0\}$ but with $7a_1 \neq 3$, the solution (39) corresponds to a 7th-order pseudo-peakon. Finally, for the specific parameter values $\{a_1=\frac{3}{7}, a_2=\frac{2}{21}, a_3=\frac{1}{105}\}$, equation (39) describes a 9th-order pseudo-peakon.

Figures 4(a)–4(f) illustrate the b-independent pseudo-peakon of Eq. (39) with the following parameter selections: $\{a_1=10, a_2=-7, a_3=1\}$, $\{a_1=1, a_2=-5, a_3=1\}$, $\{a_1=1, a_2=1, a_3=1\}$, $\{a_1=\frac{1}{2}, a_2=\frac{1}{5}, a_3=\frac{1}{10}\}$, $\{a_1=\frac{3}{7}, a_2=\frac{2}{21}, a_3=\frac{1}{105}\}$, and $\{a_1=\frac{1}{2}, a_2=\frac{1}{5}, a_3=-2\}$, respectively. Meanwhile, figures 4(g) and 4(h) demonstrate that the 9th-order pseudo-peakon solution (39) with $\{a_1=\frac{3}{7}, a_2=\frac{2}{21}, a_3=\frac{1}{105}\}$ exhibits continuous derivatives of v up to the eighth order.

The second solution of Eq. (38), corresponding to $A=0$, $B\neq 0$, and $C=0$, is associated with the b-independent peakon solution of Eq. (36),

$$v = \frac{c}{3493276440}(4120035|x-ct|^4 + 65760592|x-ct|^3 + 490916844|x-ct|^2 + 1972076400|x-ct| + 3493276440)e^{-|x-ct|}. \tag{40}$$

The third solution of Eq. (38), for $A\neq 0$, $B\neq 0$, and $C=0$, is related to the b-dependent peakon solution of Eq. (36). For simplicity, we present only an approximate form of the b-dependent peakon solution,

$$v = c(a_0 + a_1|x-ct| + a_2|x-ct|^2 + a_3|x-ct|^3 + a_4|x-ct|^4)e^{-|x-ct|}, \tag{41}$$

where the coefficients are defined as

$$\begin{aligned}
a_1 \approx\ &(2.888b^2 + 0.1728b^3 - 1.979b + 78.94)\frac{a_0^2}{10^3b^2} + (5.597b^2 + 1.06b - 1.579)\frac{a_0}{10b^2} - 0.104\frac{1}{b} + 0.07894\frac{1}{b^2},\\
a_2 \approx\ &(3.912b^2 + 0.234b^3 - 2.681b + 106.9)\frac{a_0^2}{10^3b^2} + (1.36b^2 + 0.3439b - 2.139)\frac{a_0}{10b^2} - 0.0317\frac{1}{b} + 0.1069\frac{1}{b^2},\\
a_3 \approx\ &(1.461b^2 + 0.08737b^3 - 1.001b + 39.92)\frac{a_0^2}{10^3b^2} + (0.1737b^2 + 0.02133b - 0.7984)\frac{a_0}{10b^2} - 0.001132\frac{1}{b} + 0.03992\frac{1}{b^2},\\
a_4 \approx\ &(1.986b^2 + 0.1188b^3 - 1.361b + 54.29)\frac{a_0^2}{10^4b^2} + (1.007b^2 - 0.2496b - 10.86)\frac{a_0}{10^3b^2} + 0.0003858\frac{1}{b} + 0.005429\frac{1}{b^2}.
\end{aligned}$$

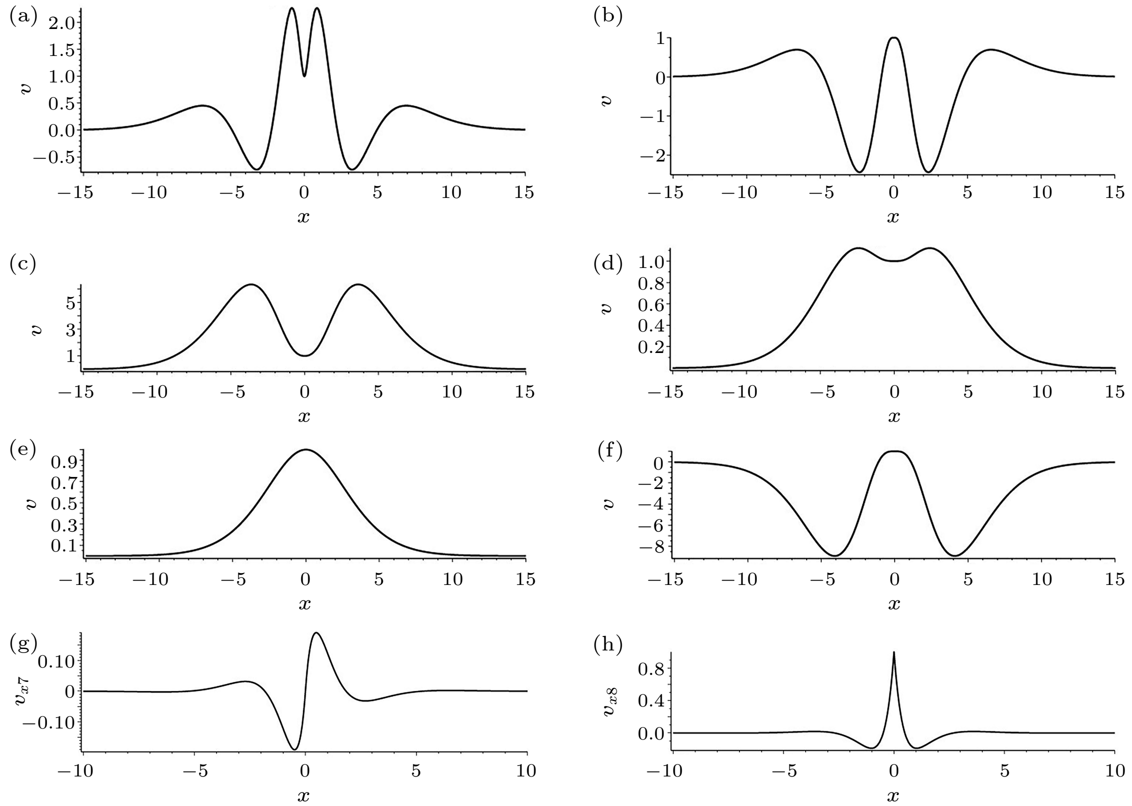

Fig. 4. (a) Pseudo-peakon with four smooth peaks expressed by Eq. (39) and $\{a_1 = 10,\ a_2 = -7,\ a_3 = 1\}$. (b) Pseudo-peakon with three smooth peaks given by Eq. (39) and $\{a_1 = 1,\ a_2 = -5,\ a_3 = 1\}$. (c) M-shaped pseudo-peakon expressed by Eq. (39) with $\{a_1 = 10,\ a_2 = 1,\ a_3 = 1\}$. (d) Pseudo-peakon with double smooth peaks expressed by Eq. (39) and $\{a_1 = \frac{1}{2},\ a_2 = \frac{1}{5},\ a_3 = \frac{1}{10}\}$. (e) Single smooth peaked pseudo-peakon expressed by Eq. (39) with $\{a_1 = \frac{3}{7},\ a_2 = \frac{2}{21},\ a_3 = \frac{1}{105}\}$. (f) W-shaped pseudo-peakon expressed by Eq. (39) with $\{a_1 = \frac{1}{2},\ a_2 = \frac{1}{5},\ a_3 = -2\}$. (g) The seventh-order derivative of v with respect to x of (e). (h) The eighth-order derivative of v with respect to x of (e).

The parameter a_0 is exactly related to b through the equation

$$(c_1b^3 + c_2b^2 - c_3b + c_4)a_0^3 + (2c_3b - c_2b^2 - 3c_4)a_0^2 + (3c_4 - c_3b)a_0 - c_4 = 0, \tag{42}$$

where the constants $\{c_1,\ c_2,\ c_3,\ c_4\}$ are fixed as

$$c_1 = 6148785446084363853888957425657644440000,$$
$$c_2 = 1027947088589364982489126801821313554216 0,$$
$$c_3 = 704527722099699612484901344671517650263 3,$$
$$c_4 = 280965775971808797424508037702679295549980.$$

It can be proven that for every real value of b, there exist one real and two conjugate complex solutions a_0 of Eq. (42). This procedure can be extended to larger values of J. Through extensive calculations, we find that for every odd integer $J = 2n + 1$ with $n \geq 1$, in addition to the b-independent pseudo-peakon solution (3), there exist one real b-independent peakon, one real b-dependent peakon, and $J - 3 = 2(n - 1)$ complex b-dependent peakons. Similarly, for every even integer $J = 2n + 2$ with $n \geq 1$, in addition to the b-independent pseudo-peakon solution (3), there exist one real b-independent peakon, two real b-dependent peakons, and $J - 4 = 2(n - 1)$ complex b-dependent peakons. These conclusions have been verified for $J \leq 14$ using MAPLE and its contained definition on the higher-order derivatives of the sign function.

7. Summary and Discussions. In summary, we have explored the rich structures of peakon and pseudo-peakon solutions for a class of higher-order b-family Eq. (2), referred to as the J-bF equations. Our primary focus has been on the conjectures 1–3 that the J-bF equation admits one b-independent pseudo-peakon solution of the form of Eq. (3), one b-independent peakon solution of the form of Eq. (8), and one b-dependent peakon solution of the form of Eq. (10). These conjectures have been analytically verified for $J \leq 14$ and/or $J \leq 9$ using the computational system MAPLE, which includes a built-in definition of the higher-order derivatives of the sign function. While a rigorous proof for arbitrary J remains an open problem, the results presented in this paper strongly support the validity of the conjectures.

The b-independent pseudo-peakon solution (3) is a 3rd-order pseudo-peakon for general arbitrary constants a_i, except for some special cases. When the first two constants a_1 and a_2 satisfy the constraint of Eq. (4), the solution (3) becomes a 5th-order pseudo-peakon solution. The 7th-order pseudo-peakon solutions are contained in Eq. (3) when the first four constants a_i $(i = 1, 2, 3, 4)$ satisfy two constraints of Eqs. (4) and (5). Furthermore, $(2n + 1)$th-order pseudo-peakons can be found by introducing $n - 1$

constraints,

$$\frac{\partial^{2i-1}v}{\partial x^{2i-1}}\Big|_{\xi\to 0^+} - \frac{\partial^{2i-1}v}{\partial x^{2i-1}}\Big|_{\xi\to 0^-} = 0,$$
$$i = 2, \ldots, n \geq 2.$$

Beyond the pseudo-peakon solutions, we have also identified the existence of both b-independent and b-dependent peakon solutions. The b-independent peakons are characterized by their independence from the parameter b, and their explicit forms have been derived for various values of J. These solutions exhibit non-smooth peaks and are distinct from the pseudo-peakons, which possess higher-order derivative discontinuities.

In addition to the b-independent peakons, we have discovered b-dependent peakon solutions, whose forms explicitly depend on the parameter b. Through detailed and intricate calculations, we have shown that for odd integers J and any real b, there exists only one real b-dependent peakon solution; whereas for even integer cases, there are two real b-dependent peakon solutions. This distinction highlights the nuanced relationship between the parameters b and J and the structure of the peakon solutions.

The existence of both b-independent and b-dependent peakons, along with the pseudo-peakon solutions, underscores the rich dynamical structure of the J-bF equations. These solutions not only generalize previous results for lower-order equations but also provide new insights into the interplay between nonlinearity and dispersion in higher-order wave models.

Future research directions in the study of peakon systems related to this paper could focus on the following areas:

Higher-Order Generalizations. Investigate all possible higher-order extensions of all known peakon systems and establish the corresponding conjectures similar to Eqs. (3), (8), and (10) for their possible peakon or pseudo-peakon solutions for all higher extensions.

Proof of Conjectures. Rigorously prove the conjectures related to these generalized peakon systems. This would involve advanced analytical techniques to validate theoretical predictions.

Interactions of Peakons and Pseudo-Peakons. Study the possible interactions between different types of peakons and pseudo-peakons. Understanding these interactions could reveal new dynamics and stability properties.

Stability of New Peakons and Pseudo-Peakons. Analyze the stability of newly discovered peakon and pseudo-peakon solutions. This includes both linear and nonlinear stability analysis to determine their robustness under different types of perturbations.

Integrable and/or Non-Integrable Properties. Explore the integrable and non-integrable properties of various generalized models and their corresponding multi-peakon dynamical systems.

Mathematical Structures and Geometric Properties. Investigate the underlying mathematical structures, including geometric properties, of the generalized models. This could involve studying the Hamiltonian structures, conservation laws, symmetries, symplectic geometry, and other geometric aspects.

Physical Applications. Explore potential physical applications of these generalized peakon models. This could include applications in fluid dynamics, nonlinear optics, and other areas where solitary waves, peakons, and pseudo-peakons play a crucial role.

These research directions would significantly advance our understanding of peakon systems and their broader implications in both mathematics and physics.

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