CHIN. PHYS. LETT. Vol. 28, No. 7 (2011) 074701

Boundary Layer Flow and Heat Transfer over an Exponentially Shrinking Sheet *

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(Received 15 April 2011)

An analysis is made to study boundary layer flow and heat transfer over an exponentially shrinking sheet. Using similarity transformations in exponential form, the governing boundary layer equations are transformed into self-similar nonlinear ordinary differential equations, which are then solved numerically using a very efficient shooting method. The analysis reveals the conditions for the existence of steady boundary layer flow due to exponential shrinking of the sheet and it is found that when the mass suction parameter exceeds a certain critical value, steady flow is possible. The dual solutions for velocity and temperature distributions are obtained. With increasing values of the mass suction parameter, the skin friction coefficient increases for the first solution and decreases for the second solution.

PACS: 47.15.Cb, 44.20.+b DOI:10.1088/0256-307X/28/7/074701

Viscous boundary layer flow due to a stretching/shrinking sheet is very significant due to its huge applications in many manufacturing processes in industry, such as the extraction of polymer sheets, paper production, hot rolling and glass-fiber production.[1,2] Crane[3] first investigated the steady boundary layer flow of an incompressible viscous fluid over a linearly stretching plate and gave an exact similarity solution in a closed analytical form. The work of Crane[3] was extended by Gupta and Gupta,[4] Chen and Char,[5] and Pavlov[6] under various physical conditions. In addition, Sankara and Watson[7] investigated the flow of micropolar fluids past a stretching sheet. Andersson and Dandapat[8] demonstrated the flow of a power-law fluid over a stretching sheet. Vajravelu[9] and Cortell[10,11] discussed boundary layer flows over a nonlinear stretching sheet.

The development of the unusual type of flow due to shrinking was first observed by Wang[12] when he was investigating the behaviour of a liquid film on an unsteady stretching sheet. Later, Miklavčič and Wang[13] established the criteria for the existence and uniqueness of the similarity solution of the equation for the flow due to a shrinking sheet and found that the flow depends on externally imposed mass suction. Hayat *et al.*[14] reported an analytic solution of magnetohydrodynamic (MHD) flow of a second grade fluid over a shrinking sheet. Hayat *et al.*[15] also obtained an analytical solution of the MHD rotating flow of a second grade fluid past a porous shrinking sheet by the homotopy analysis method (HAM). In another paper, Hayat *et al.*[16] discussed the mass transfer in a steady two-dimensional MHD boundary layer flow of an upper-convected Maxwell fluid past a porous shrinking sheet in the presence of a chemical reaction, and the expressions for the velocity and the concentration distributions were obtained using HAM. Fang and Zhang[17] found a closed-form exact solution for two-dimensional MHD flow over a porous shrinking sheet subjected to wall mass transfer. Noor *et al.*[18] reported a series solution of MHD viscous flow due to a shrinking sheet using the Adomian decomposition method (ADM). Fang *et al.*[19] solved analytically the viscous flow over a porous shrinking sheet with a second order slip flow model. Cortell[20] discussed the MHD viscous flow caused by a shrinking sheet with suction for two-dimensional and axisymmetric cases. Fang *et al.*[21] studied unsteady viscous flow over a shrinking surface with mass suction. The unsteady boundary layer flow of an electrically conducting fluid on a shrinking surface with a constant transverse magnetic field was investigated by Merkin and Kumaran.[22] Wang[23] investigated the stagnation-point flow towards a shrinking sheet for both two-dimensional and axisymmetric cases. Wang's[23] work was extended by Ishak *et al.*,[24] Bhattacharyya and Layek,[25] and Bhattacharyya *et al.*[26]

Over the last few decades, in almost all investigations on the flow past a stretching sheet, the flow occurs because of the linear stretching velocity of the flat sheet. However, the boundary layer flow induced by an exponentially stretching/shrinking sheet is not studied much, though it is very important and realistic flow frequently appears in many engineering processes. Magyari and Keller[27] first considered the boundary layer flow due to an exponentially stretching sheet and he also investigated the heat transfer in the flow taking an exponentially varying wall temperature. Elbashbeshy[28] numerically examined the flow and heat transfer over an exponentially stretching surface considering wall mass suction. Khan and Sanjayanand[29] studied the flow of viscoelastic fluid

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and heat transfer over an exponentially stretching sheet with viscous dissipation effect. Partha *et al.*[30] obtained a similarity solution for mixed convection flow past an exponentially stretching surface by taking into account the influence of viscous dissipation on the convective transport. Sanjayanand and Khan[31] discussed the effects of heat and mass transfer on the boundary layer flow of viscoelastic fluid. Al-Odat *et al.*[32] explained the effect magnetic field on thermal boundary layer on an exponentially stretching continuous surface with an exponential temperature distribution. Recently, Sajid and Hayat[33] showed the influence of thermal radiation on the boundary layer flow past an exponentially stretching sheet and they reported a series solutions for velocity and temperature using HAM.

However, the flow dynamics due to an exponentially shrinking sheet is still unknown. Thus, in this Letter, we investigate the boundary layer flow and heat transfer over an exponentially shrinking sheet. Using an exponential form of similarity transformation, the governing equations are transformed into self-similar ordinary differential equations. Then those nonlinear self-similar equations are solved numerically using the shooting method and the flow characteristics are discussed.

Consider the steady two-dimensional boundary layer flow of a Newtonian fluid and heat transfer over an exponentially shrinking sheet. The governing equations of motion and the energy equation may be written in usual notation as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

where u and v are the velocity components in x- and y-directions, respectively, $\upsilon(=\mu/\rho)$ is the kinematic fluid viscosity, ρ is the fluid density, μ is the coefficient of fluid viscosity, T is the temperature, κ is the fluid thermal conductivity and c_p is the specific heat.

The boundary conditions are given by

$$u=U_w(x),\ \ v=v_w \text{ at } y=0;\ \ u\to 0 \text{ as } y\to ß,nfty, \tag{4}$$

$$T=T_w=T_\infty+T_0\exp(\tfrac{x}{2L}) \text{ at } y=0;\quad T\to T_\infty \text{ as } y\to\infty, \tag{5}$$

where T_w is the variable temperature at the sheet, T_∞ is the free stream temperature assumed to be constant and T_0 is a constant which measures the rate of temperature increase along the sheet. The shrinking velocity U_w is given by

$$U_w(x)=-c\exp(\tfrac{x}{L}), \tag{6}$$

where $c>0$ is shrinking constant.

Here v_w is the variable wall mass transfer velocity given by

$$v_w=v_0\exp(\tfrac{x}{2L}), \tag{7}$$

where v_0 is a constant with $v_0<0$ for mass suction and $v_0>0$ for mass injection.

Now, the stream function $\psi(x,y)$ is introduced as

$$u=\frac{\partial\psi}{\partial y} \text{ and } v=-\frac{\partial\psi}{\partial x}. \tag{8}$$

For relations in Eq. (8), the continuity equation (1) is identically satisfied, the momentum equation (2) and the temperature equation (3) are reduced to the forms

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\upsilon\frac{\partial^3\psi}{\partial y^3}, \tag{9}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}=\frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2}. \tag{10}$$

The boundary conditions in Eq. (4) for the velocity components become

$$\frac{\partial\psi}{\partial y}=U_w(x),\ \frac{\partial\psi}{\partial x}=-v_w \text{ at } y=0;\quad \frac{\partial\psi}{\partial y}\to 0 \text{ as } y\to\infty. \tag{11}$$

The dimensionless variables for ψ and T read[29,31]

$$\psi=\sqrt{2\upsilon Lc}\,f(\eta)\exp(\tfrac{x}{2L}), T=T_\infty+(T_w-T_\infty)\theta(\eta)\,, \tag{12}$$

where η is the similarity variable defined as $\eta=y\sqrt{\frac{c}{2\upsilon L}}\exp(\frac{x}{2L})$.

Using the relations in Eq. (12) we finally obtain the nonlinear self-similar equations

$$f'''+ff''-2f'^2=0, \tag{13}$$

$$\theta''+Pr(f\theta'-f'\theta)=0, \tag{14}$$

where $Pr=\mu c_p/\kappa$ is the Prandtl number.

The boundary conditions (11) and (5) reduce to the forms

$$f(\eta)=S, f'(\eta)=-1 \text{ at } \eta=0;\quad f'(\eta)\to 0 \text{ as } \eta\to\infty, \tag{15}$$

$$\theta(\eta)=1 \text{ at } \eta=0;\quad \theta(\eta)\to 0 \text{ as } \eta\to\infty, \tag{16}$$

where $S=-v_0/\sqrt{\frac{\upsilon c}{2L}}$ is the mass transfer parameter. $S>0$ $(v_0<0)$ corresponds mass suction and $S<0$ $(v_0>0)$ corresponds to the mass injection.

The nonlinear coupled differential Eqs. (13) and (14), along with the boundary conditions (15) and (16), form a two-point boundary value problem (BVP) and is solved using the shooting method, by converting it into an initial value problem (IVP). In this method we have to choose a suitable finite value of $\eta \to \infty$, say η_∞. We set the following first-order system

$$f' = p,\ p' = q,\ q' = 2p^2 - fq, \tag{17}$$

$$\theta' = z,\ z' = -Pr(fz - p\theta), \tag{18}$$

under the boundary conditions

$$f(0) = S,\ p(0) = -1,\ \theta(0) = 1. \tag{19}$$

To solve Eqs. (17) and (18) with Eq. (19) as an IVP we need the values for $q(0)$ i.e. $f''(0)$ and $z(0)$, i.e. $\theta'(0)$ but no such values are given. The initial guess values for $f''(0)$ and $\theta'(0)$ are chosen and the fourth order Runge–Kutta method is applied to obtain a solution. Then we compare the calculated values of $f'(\eta)$ and $\theta(\eta)$ at $\eta_\infty(= 40)$ under the given boundary conditions $f'(\eta_\infty) = 0$ and $\theta(\eta_\infty) = 0$ and adjust the values of $f''(0)$ and $\theta'(0)$ using the Secant method to give abetter approximation for the solution. The step-size is taken as $\Delta\eta = 0.01$. The process is repeated until we obtain results correct up to the desired accuracy of the 10^{-7} level, which fulfills the convergence criterion.

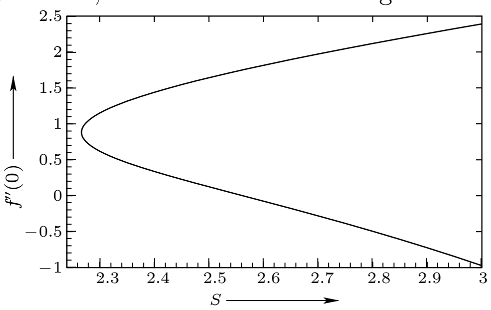

Fig. 1. Skin friction coefficient $f''(0)$ for various values of S.

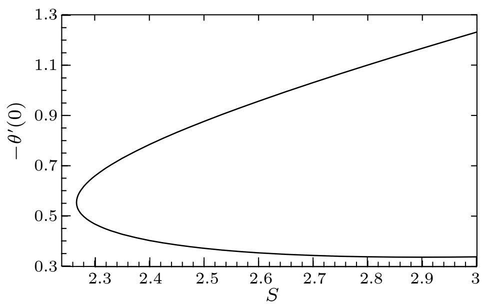

Fig. 2. Temperature gradient at the sheet $-\theta'(0)$ for various values of S.

Numerical computations are carried out for obtaining the condition under which the steady flow over an exponentially shrinking sheet is possible. The shrinking rate in the exponential case is much faster than that of the linear case. Thus, the amount of vorticity generated due to exponential shrinking is greater than that of linear shrinking. The flow due to linear shrinking was studied by Miklavčič and Wang,[13] and Fang and Zhang[17] (with MHD effect). According to their analyses the steady two-dimensional flow of Newtonian fluids due to a shrinking sheet with wall mass transfer will appear only when the mass suction parameter is greater than or equal to 2. However, for the exponential shrinking flow that amount of mass suction is not sufficient for steady flow, from this investigation it is found that if the mass suction parameter is greater than or equal to 2.266684, then only the steady flow due to an exponentially shrinking sheet is possible. This is compatible with the physics of the flow. Hence, to keep the larger amount of vorticity generated due to exponential shrinking inside the boundary layer flow needs more mass suction than the linear case. Thus, for an exponentially shrinking sheet, the similarity solution exists when the mass suction parameter S satisfies the condition $S \geq 2.266684$ and consequently for $S < 2.266684$ the flow has no similarity solution. Moreover, for $S \geq 2.266684$ dual similarity solutions are obtained. In this regard, the values of the skin friction coefficient $f''(0)$ for different values of S are depicted in Fig. 1. The values of temperature gradient at the sheet $-\theta'(0)$ which are proportional to the rate of heat transfer from the sheet are plotted in Fig. 2 for different values of the mass suction parameter. Furthermore, from those two figures it is also observed that the skin friction coefficient $f''(0)$ increases with S for the first solution and it decreases for the second solution. The value of $-\theta'(0)$ shows the nature similar to the skin friction coefficient for the first solution, but for the second though it initially decreases with increasing S, for large values of S it starts to increase again.

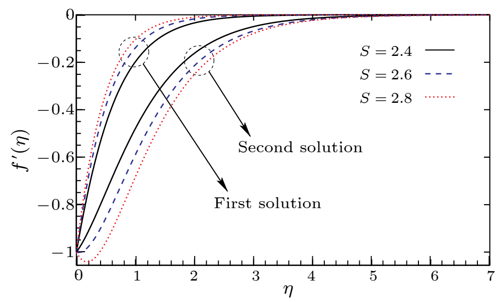

Fig. 3. Dual velocity profiles $f'(\eta)$ for various values of S.

The variations in velocity profiles and shear stress profiles for different values of mass suction parameter S are demonstrated in Figs. 3 and 4, respectively. It is seen that the momentum boundary layer thickness for the first solution is always thinner than that of the second solution. Figure 3 also shows that the dimensionless velocity profile $f'(\eta)$ increases with the increasing values of S for first solution and the velocity decreases with S for the second solution. On the other hand, for the first solution the value of the shear stress profile at first increases with increasing S, but for large η it decreases and reverse nature is noticed for the case of the second solution. In Fig. 5, the dual temperature profiles $\theta(\eta)$ are exhibited for various values of S. The temperature at a point decreases for an increase of S for the first solution, while for the second solution the temperature increases with S. Similar to the velocity field, the thermal boundary layer thickness for the second solution is thicker than that of the first solution.

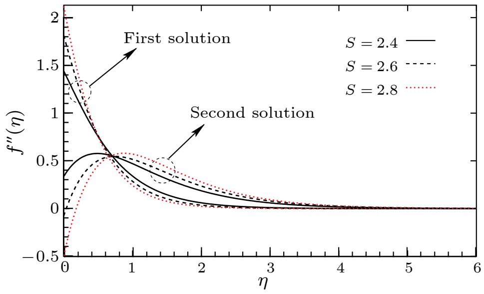

Fig. 4. Shear stress profiles $f''(\eta)$ for various values of S.

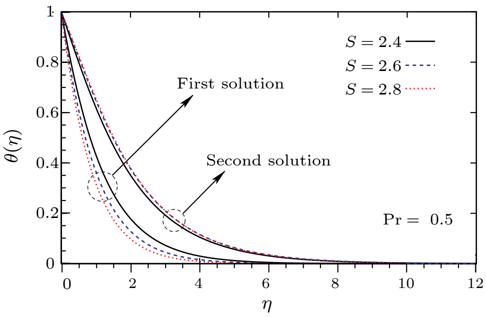

Fig. 5. Dual temperature profiles $\theta(\eta)$ for various values of S.

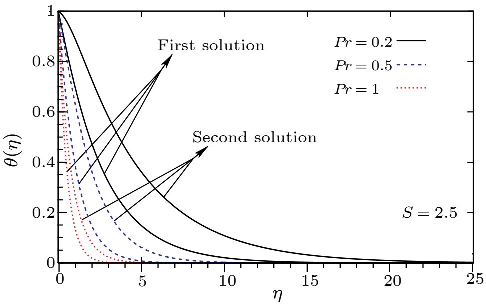

Fig. 6. Dual temperature profiles $\theta(\eta)$ for several values of Pr.

Finally, the effects of the Prandtl number Pr on the dimensionless temperature profiles are presented in Fig. 6. For both solutions, the temperature at a point is found to decrease with the increasing Pr. Also, the thermal boundary layer thickness reduces significantly due to increase of Pr for both the cases. The Prandtl number is inversely proportional to the thermal conductivity. Thus the fluids with the lower Prandtl number have higher thermal conductivities and consequently the heat diffusion is faster in this case. On the other hand, for higher Pr fluids the heat diffusion slows down.

The boundary layer flow and heat transfer over an exponentially shrinking sheet is investigated. The similarity equations are obtained and solved numerically by the shooting method. The study reveals that the steady flow due to an exponentially shrinking sheet is possible only when the mass suction parameter $S \geq 2.266684$, and dual similarity solutions for velocity field and temperature distribution are found. In addition, for the first solution the velocity increases with mass suction and decreases for the second solution. The opposite behavior is observed in the temperature distribution for increment of suction. For both solutions, the thermal boundary layer thickness becomes thinner due to the increasing Prandtl number.

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