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Rogue Wave, Breathers and Bright-Dark-Rogue Solutions for the Coupled Schrödinger Equations

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We construct explicit rogue wave solutions, breather solitons, and rogue-bright-dark solutions for the coupled non-linear Schrödinger equations by the Darboux transformation.

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The coupled non-linear Schrödinger (CNLS/NLS) equations are a significant model in areas such as nonlinear optics,[1] bio-physics,[2] multi-component Bose–Einstein condensates at zero temperature,[3] finance[4] and oceanics.[5] However, they become integrable for a special choice of the parameter (Manakov[6] model). On the other hand, the Manakov model has appeared in a Kerr-type approximation of photorefractive crystals. Therefore this equation can explicitly describe some physical phenomena. There is lots of the research about these kinds of system.[7−9]

In this Letter, we consider an integrable coupled Schrödinger equation

$$\mathrm{i}q_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2)q_1 = 0, \tag{1a}$$
$$\mathrm{i}q_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2)q_2 = 0. \tag{1b}$$

The integrability of this system was proved by Manakov, which is here named as the integrable Manakov[6] system. Recently, the rogue wave phenomenon was popular in nonlinear science. Rogue waves are strong wavelets that may appear in the ocean when special conditions are met. These waves can be two, three or even more times higher than the average wave crest. Rogue waves are also known as freak waves, monster waves, killer waves, giant waves or extreme waves.[10] The first-order rational solution for the nonlinear Schrödinger equation was given by Peregrine in 1983.[11] The rogue wave of the NLS equation was derived by Darboux transformation.[12] Recently, we constructed some general rogue wave solutions by the Darboux–Liouville transformation.[13] It is well known that there have been lots of results about the rogue wave for nonlinear Schrödinger equations. However, to the best of our knowledge, there are no reports on exact solutions related to the rogue wave solution of the coupled Schrödinger equation. Here we also obtain the new breather solutions and bright-dark-rogue solutions as a by-product.

The Darboux transformation[14−17] is a powerful method to construct some interesting solutions (solitons, positons, rogue wave solution) in the integrable system. The goal of this work is to use the Darboux transformation to construct the rogue wave solution, breathers and bright-dark-rogue solution. These special solutions may play an important role in the research of some physical phenomena such as BEC or nonlinear optics.

Firstly, we construct the rogue wave solution by the Darboux transformation. The system (1) admits the following Lax pair

$$\Phi_x = U\Phi = [\mathrm{i}\zeta U_1 + U_0]\Phi, \tag{2a}$$
$$\Phi_t = V\Phi = [3\mathrm{i}\zeta^2 U_1 + 3\zeta U_0 + \mathrm{i}\sigma_3(U_{0,x} - U_0^2)]\Phi, \tag{2b}$$

where

$$U_0 = \begin{pmatrix} 0 & q_1 & q_2 \\ -q_1^* & 0 & 0 \\ -q_2^* & 0 & 0 \end{pmatrix},$$

$U_1 = \mathrm{diag}\{-2, 1, 1\}$, $\sigma_3 = \mathrm{diag}\{1, -1, -1\}$, and Φ is a 3×3-matrix valued eigenfunction, ζ is a spectral parameter. The compatibility condition of the Lax pair (2) gives the CNLS (1). It is well known that the Darboux transformation[7,15,16] for this system (2) is

$$\mathrm{T} = \zeta I - [\zeta_1^* - (\zeta_1^* - \zeta_1)P_1], \quad P_1 = \frac{\Phi_1\Phi_1^\dagger}{\Phi_1^\dagger\Phi_1}, \tag{3}$$

where $\Phi_1 = \Phi(x,t,\zeta_1)(m_1, m_2, m_3)^T$, m_1, m_2, m_3 are constants, I is a 3×3 identity matrix and $\Phi(x,t;\zeta_1)$ is the fundamental solution for the Lax pair eqaution at $\zeta = \zeta_1$, $q_i = q_i[0]$ $(i = 1, 2)$. It follows the transformation between the fields,

$$q_1[1] = q_1[0] + 6\mathrm{Im}(\zeta_1)\frac{\phi_1\phi_2^*}{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2},$$
$$q_2[1] = q_2[0] + 6\mathrm{Im}(\zeta_1)\frac{\phi_1\phi_3^*}{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2}, \tag{4}$$

where $(\phi_1,\ \phi_2,\ \phi_3)^T = \Phi_1$.

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We start from the seed solution-nonzero plane wave solution. We denote them as

$$q_1[0] = c_1 \exp[\mathrm{i}\theta_1], \quad q_2[0] = c_2 \exp[\mathrm{i}\theta_2], \tag{5}$$

where $\theta_i = d_i x + (2c_1^2 + 2c_2^2 - d_i^2)t$ with c_i and d_i $(i = 1, 2)$ being arbitrary constants. However, these Lax pair equations are variable coefficient differential equations. We can not directly solve them. Luckily we can convert this system into constant coefficient differential equations by a gauge transformation. In the following, we give the transformation for the Lax pair equations (2) at $q_1 = q_1[0]$ and $q_2 = q_2[0]$. We can obtain

$$\begin{aligned}\Psi_x &= (MUM^{-1} + M_x M^{-1})\Psi = U_1\Psi,\\ \Psi_t &= (MVM^{-1} + M_t M^{-1})\Psi = V_1\Psi,\end{aligned} \tag{6}$$

where $\Psi = M\Phi$, $M = \mathrm{diag}\{\exp[-\frac{\mathrm{i}}{3}(\theta_1 + \theta_2)], \exp[\frac{\mathrm{i}}{3}(2\theta_1 - \theta_2)], \exp[\frac{\mathrm{i}}{3}(2\theta_2 - \theta_1)]\}$. Simply computing, we have

$$U_1 = \begin{pmatrix} \chi_{11} & c_1 & c_2 \\ -c_1 & \chi_{22} & 0 \\ -c_2 & 0 & \chi_{33} \end{pmatrix},$$

$$\begin{aligned} V_1 &= \mathrm{i}U_1^2 - [\frac{2}{3}(d_1 + d_2) - 2\zeta_1]U_1 + mI, \\ \chi_{11} &= -2\mathrm{i}\zeta_1 - \frac{\mathrm{i}}{3}(d_1 + d_2), \\ \chi_{22} &= \mathrm{i}\zeta_1 + \frac{\mathrm{i}}{3}(2d_1 - d_2), \\ \chi_{33} &= \mathrm{i}\zeta_1 + \frac{\mathrm{i}}{3}(2d_2 - d_1), \end{aligned} \tag{7}$$

where $m = 2\mathrm{i}\zeta_1^2 + \frac{2\mathrm{i}}{3}(c_1^2 + c_2^2) + \frac{2\mathrm{i}}{9}(d_1^2 - d_1 d_2 + d_2^2) + \frac{2\mathrm{i}\zeta_1}{3}(d_1 + d_2)$. In Refs. [7–9], the authors consider the solution when the characteristic equation for U_1 has a single and double root. Indeed, the characteristic equation for U_1 can possess a triple root. Our work is looking for the solution for this special assumption. Equations (6) are the constant coefficient differential equations. In order to look for the "rational solution", we choose the parameters to satisfy $\alpha = d_2 + 3e_1$, $e_1 = \mathrm{Re}(\zeta_1)$, $d_1 = d_2 - 2\alpha, c_1 = \pm 2\alpha$, $c_2 = \pm 2\alpha$, $\mathrm{Im}(\zeta_1) = \pm\sqrt{3}\alpha$, d_2 and $\mathrm{Re}(\zeta_1)$ are arbitrary real numbers. The matrix U_1 cannot diagonalize and can merely decompose with the Jordan form with the diagonal element. Therefore the fundamental-solution matrix for Lax pair equations (2) at $\zeta = \zeta_1$, $q_i = q_i[0]$ $(i = 1, 2)$ are $\Phi(x, t; \zeta_1) = M^{-1}\Theta$,

$$\Theta = \begin{pmatrix} \varpi_{11} & 4\alpha^2\gamma + 2\sqrt{3}\alpha & 4\alpha \\ \varpi_{12} & -2\alpha^2(\sqrt{3} - \mathrm{i})\gamma - 2\alpha & -2\alpha^2(\sqrt{3} - \mathrm{i}) \\ \varpi_{13} & -2\alpha^2(\sqrt{3} + \mathrm{i})\gamma - 2\alpha & -2\alpha^2(\sqrt{3} + \mathrm{i}) \end{pmatrix},$$

$$\begin{aligned} \varpi_{11} &= 4\alpha^2(\gamma^2 + 2\mathrm{i}t) + 4\sqrt{3}\alpha\gamma + 2, \\ \varpi_{12} &= -2(\sqrt{3} - \mathrm{i})\alpha^2(\gamma^2 + 2\mathrm{i}t) - 4\alpha\gamma, \\ \varpi_{13} &= -2(\sqrt{3} + \mathrm{i})\alpha^2(\gamma^2 + 2\mathrm{i}t) - 4\alpha\gamma, \end{aligned}$$

where $\gamma = x + 2\sqrt{3}\beta\mathrm{i}t$, and $\beta = \alpha - \mathrm{i}\sqrt{3}e_1$. In what follows, we use the Darboux transformation (4) to derive the rational solution for CNLS (1).

(1) *Rogue wave* I. Choosing the parameters $m_1 = 0$, $m_2 = 1$, $m_3 = 0$, we obtain the one rogue wave I by the formula (4)

$$\begin{aligned} q_1[1] &= \alpha\Big[-1 - \mathrm{i}\sqrt{3} \\ &+ \frac{-6\delta\alpha\sqrt{3} - 36t\alpha^2\sqrt{3} - 3 + \mathrm{i}(36\alpha^2 t + 6\delta\alpha + 5\sqrt{3})}{12\alpha^2\delta^2 + 8\delta\alpha\sqrt{3} + 144t^2\alpha^4 + 5}\Big] \\ &\cdot \exp[\mathrm{i}\theta_1], \\ q_2[1] &= \alpha\Big[-1 + \mathrm{i}\sqrt{3} \\ &+ \frac{-6\delta\alpha\sqrt{3} + 36t\alpha^2\sqrt{3} - 3 + \mathrm{i}(36\alpha^2 t - 6\delta\alpha - 5\sqrt{3})}{12\alpha^2\delta^2 + 8\delta\alpha\sqrt{3} + 144\alpha^4 t^2 + 5}\Big] \\ &\cdot \exp[\mathrm{i}\theta_2], \end{aligned} \tag{8}$$

where $\delta = x + 6e_1 t$. This kind of rogue wave solution is similar to the first-order rogue wave solution for the nonlinear Schrödinger equation.[12]

(2) *Rogue wave* II. Choosing the parameters $m_1 = 1$, $m_2 = 0$, $m_3 = 0$, we obtain the rogue wave II by the formula

$$\begin{aligned} q_1[1] &= \alpha[-1 - \mathrm{i}\sqrt{3} + \frac{G_1 + \mathrm{i}H_1}{D}]\exp[\mathrm{i}\theta_1], \\ q_2[1] &= \alpha[-1 + \mathrm{i}\sqrt{3} + \frac{G_2 + \mathrm{i}H_2}{D}]\exp[\mathrm{i}\theta_2], \end{aligned} \tag{9}$$

where

$$\begin{aligned} D &= 1728\alpha^8 t^4 + 288\alpha^6\delta^2 t^2 + 384\sqrt{3}\alpha^5\delta t^2 + 12\alpha^4\delta^4 \\ &\quad + 432\alpha^4 t^2 + 16\sqrt{3}\alpha^3\delta^3 + 24\alpha^2\delta^2 + 4\sqrt{3}\alpha\delta + 1, \\ G_1 &= -864\sqrt{3}\alpha^6 t^3 - 144\sqrt{3}\alpha^5\delta t^2 - 72\sqrt{3}\alpha^4\delta^2 t \\ &\quad - 216\alpha^4 t^2 - 12\sqrt{3}\alpha^3\delta^3 - 144\alpha^3\delta t - 18\alpha^2\delta^2 \\ &\quad - 12\sqrt{3}\alpha^2 t + 3, \\ G_2 &= +864\sqrt{3}\alpha^6 t^3 - 144\sqrt{3}\alpha^5\delta t^2 + 72\sqrt{3}\alpha^4\delta^2 t \\ &\quad - 216\alpha^4 t^2 - 12\sqrt{3}\alpha^3\delta^3 + 144\alpha^3\delta t - 18\alpha^2\delta^2 \\ &\quad + 12\sqrt{3}\alpha^2 t + 3, \\ H_1 &= +864\alpha^6 t^3 + 144\alpha^5\delta t^2 + 72\alpha^4\delta^2 t + 312\sqrt{3}\alpha^4 t^2 \\ &\quad + 12\alpha^3\delta^3 + 96\sqrt{3}\alpha^3\delta t + 18\sqrt{3}\alpha^2\delta^2 + 108\alpha^2 t \\ &\quad + 12\alpha\delta + \sqrt{3}, \\ H_2 &= +864\alpha^6 t^3 - 144\alpha^5\delta t^2 + 72\alpha^4\delta^2 t - 312\sqrt{3}\alpha^4 t^2 \\ &\quad - 12\alpha^3\delta^3 + 96\sqrt{3}\alpha^3\delta t - 18\sqrt{3}\alpha^2\delta^2 + 108\alpha^2 t \\ &\quad - 12\alpha\delta - \sqrt{3}. \end{aligned}$$

Choosing the right parameter, the pictures for $|q_1[1]|^2$ and $|q_2[2]|^2$ are shown in Fig. 1. They have different behaviors to rogue waves of NLS.[12] Indeed we can combine the three vector solution to obtain the new rogue wave, but we cannot obtain any new kinds of solution except the above two kinds. The rogue wave we obtained is different from Ref. [4] by a symmetry analysis, which is nothing but the rogue wave of NLS at different altitude.

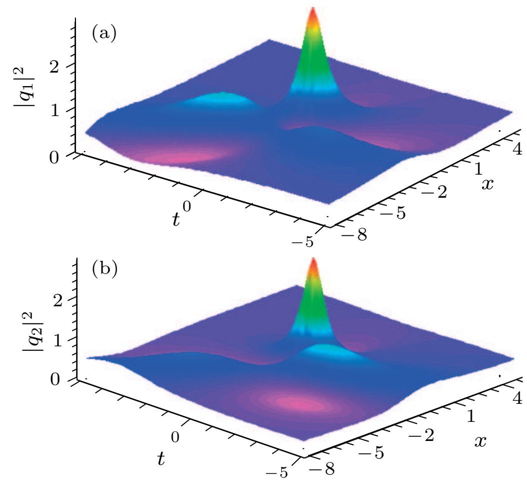

Fig. 1. Different behavior of $|q_1|^2$ and $|q_2|^2$ besides the altitude for the parameters $e_1 = \frac{1}{10}$, $d_1 = 0$.

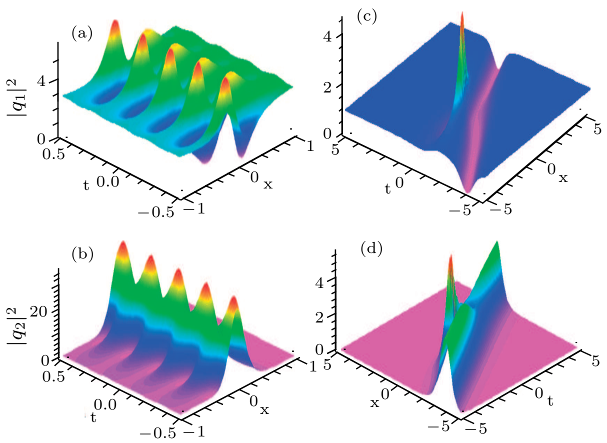

Fig. 2. (a) and (b)] with parameters $c = -2, d = 0, \zeta_1 = 2\mathrm{i}, \lambda_1 = 1 - \sqrt{5}, \tau_1 = \tau_2 = \kappa_1 = \kappa_2 = 0$; and the bright-dark-rogue solution [(c) and (d)] with parameters $c = d = 1$, $c_1 = c_2 = c_3 = 1$.

We can easily obtain the following relation for those rogue wave solutions,

$$\int_{-\infty}^{\infty} (|q_1[1]|^2 - 4\alpha^2)\mathrm{d}x = 0,$$
$$\int_{-\infty}^{\infty} (|q_2[1]|^2 - 4\alpha^2)\mathrm{d}x = 0. \qquad (10)$$

This means that $|q_1[1]|^2$ and $|q_2[1]|^2$ are equally above and below the background.

Moreover, we consider another interesting case: one is a zero solution and the other is a plane wave solution. It is interesting that we could obtain breather solutions, a bright-dark solution and a bright-dark-rogue solution by this choice. In what follows, we start from the seed solution $q_1[0] = c\exp[\mathrm{i}\theta]$, $q_2[0] = 0$, where $\theta = dx + (2c^2 - d^2)t$. Similarly, we take $M = \mathrm{diag}\{1, \exp[\mathrm{i}\theta], 1\}$. Then we can obtain the following results

$$U_1 = \begin{pmatrix} -2\mathrm{i}\zeta_1 & c & 0 \\ -c & \mathrm{i}\zeta_1 + \mathrm{i}d & 0 \\ 0 & 0 & \mathrm{i}\zeta_1 \end{pmatrix},$$
$$V_1 = \mathrm{i}U_1^2 + 2\zeta_1 U_1 + 2\mathrm{i}(\zeta_1^2 + c^2) - \mathrm{diag}\{0, 0, 2\mathrm{i}c^2\}.$$

The characteristic equation for U_1 is $(\lambda - \mathrm{i}\zeta_1)[(\lambda + 2\mathrm{i}\zeta_1)(\lambda - \mathrm{i}\zeta_1 - \mathrm{i}d) + c^2] = 0$. Then we would classify the characteristic equation into two kinds that different solution behaviors will exhibit.

Case I: *single root*. Assume that the characteristic equation of matrix U_1 has three distinct roots. Denote them as $\mathrm{i}\zeta_1$, $\lambda_1 = \frac{\mathrm{i}}{2}[-\zeta_1 + d + \sqrt{9\zeta_1^2 + 6\zeta_1 d + d^2 + 4c^2}]$, $\lambda_2 = \mathrm{i}(d - \zeta_1) - \lambda_1$. Finally, we obtain the fundamental solution for the Lax pair equations $\Phi(x, t; \zeta_1) = M^{-1}Q\Omega$,

$$Q = \begin{pmatrix} 0 & c & \lambda_1 + 2\mathrm{i}\zeta_1 \\ 0 & \lambda_1 + 2\mathrm{i}\zeta_1 & c \\ 1 & 0 & 0 \end{pmatrix},$$
$$\Omega = \begin{pmatrix} e^{\mathrm{i}\zeta_1 x + 3\mathrm{i}\zeta_1^2 t} & 0 & 0 \\ 0 & \kappa_{22} & 0 \\ 0 & 0 & \kappa_{33} \end{pmatrix},$$

$$\kappa_{22} = \exp\{\lambda_1 x + [\mathrm{i}\lambda_1^2 + 2\zeta_1\lambda_1 + 2\mathrm{i}(\zeta_1^2 + c^2)]t\},$$
$$\kappa_{33} = \exp\{\lambda_2 x + [\mathrm{i}\lambda_2^2 + 2\zeta_1\lambda_2 + 2\mathrm{i}(\zeta_1^2 + c^2)]t\}.$$

Substituting the solution to the formula (4) and taking the parameters $m_1 = 1$, $m_2 = e^{\tau_1 + \mathrm{i}\kappa_1}$, $m_3 = e^{\tau_2 + \mathrm{i}\kappa_2}$, ($\tau_i$, κ_i $(i = 1, 2)$ are real numbers), we could obtain the breather solutions for the CNLS equation. This kind of solution is a periodic function or a quasi-periodic function. Taking the proper parameters, then Figs. 2(a) and 2(b) exhibit the dynamical behavior of the solutions $q_1[1]$ and $q_2[1]$. When $\mathrm{Re}(\zeta_1) = \mathrm{Im}(\lambda_1) = d = 0$, the solution is a time periodic function and the period is $2\pi/|d^2 - \mathrm{Im}(\zeta_1)^2|$.

Case II: *multiple roots*. Assuming $\zeta_1 = -\frac{1}{3}(d + 2\mathrm{i}c)$, the characteristic equation for U_1 has a double root. It follows that the fundamental solution for the Lax pair equations at $\zeta_1 = -\frac{1}{3}(d + 2\mathrm{i}c)$ is $\Phi(x, t; \zeta_1) = M^{-1}Q\Omega$,

$$Q = \begin{pmatrix} 0 & -c & 1 \\ 0 & -c & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\Omega = \begin{pmatrix} 0 & 0 & \exp(\eta_1) \\ \exp(\eta_2) & [x - 2(d + \mathrm{i}c)t]\exp(\eta_2) & 0 \\ 0 & \exp(\eta_2) & 0 \end{pmatrix},$$

$$\eta_1 = \frac{1}{3}(2c - \mathrm{i}d)x + \frac{1}{3}[-4cd + \mathrm{i}(d^2 - 4c^2)]t,$$
$$\eta_2 = \frac{1}{3}(-c + 2\mathrm{i}d)x + \frac{1}{3}[2cd + \mathrm{i}(5c^2 - 2d^2)]t.$$

Substituting the solution into the formula (4), we could obtain the bright-dark soliton and bright-dark-rogue solution. If we choose the parameter $c_3 = 0$, it follows that $q_1[1]$ is the rogue wave solution for the NLS equation and $q_2[1] = 0$. In the following, we consider the other two non-trivial cases.

Case 1: Taking the parameters $m_1 = 1$, $m_2 = 0$, $m_3 = a_3 + b_3\mathrm{i}$, we obtain the dark-bright solitons,

$$\begin{aligned} q_1[1] =& c\tanh(\frac{3}{2}\eta_{1r} + \mu)\exp(\mathrm{i}\theta), \\ q_2[1] =& -4c^2(a_3 + \mathrm{i}b_3)\mathrm{sech}(\frac{3}{2}\eta_{1r} + \mu) \\ &\cdot \exp[\mathrm{i}(dx + (3c^2 - d^2)t)], \end{aligned}$$

where $\eta_{1r} = \frac{2c}{3}(x - 2dt)$ and $\mu = \frac{1}{2}\ln(\frac{a_3^2+b_3^2}{2c^2})$. It is interesting that we can obtain the bright-dark solition solution in the focusing behavior and anomalous dispersion fibers.

Case 2: Taking parameters $m_1 = a_1 + b_1\mathrm{i}$, $m_2 = 1$, $m_3 = a_3 + b_3\mathrm{i}$, we have

$$\begin{aligned} q_1[1] =& c\Big\{1 - 4[\alpha_1^2 + \alpha_2^2 - \alpha_1 + \mathrm{i}\alpha_2]\exp(-\eta_{1r}) \\ &\cdot \Big\{[2(\alpha_1^2 + \alpha_2^2) - 2\alpha_1 + 1]\exp(-\eta_{1r}) \\ &+ (a_3^2 + b_3^2)\exp(2\eta_{1r})\Big\}^{-1}\Big\}\exp(\mathrm{i}\theta), \\ q_2[1] =& -4c\big[a_3(\alpha_1 - 1) - b_3\alpha_2 + \mathrm{i}[a_3\alpha_2 + b_3(\alpha_1 - 1)]\big] \\ &\cdot \exp[\frac{1}{2}\eta_{1r} + \mathrm{i}(dx + (3c^2 - d^2)t)] \\ &\cdot \Big\{[2(\alpha_1^2 + \alpha_2^2) - 2\alpha_1 + 1]\exp(-\eta_{1r}) \\ &+ (a_3^2 + b_3^2)\exp(2\eta_{1r})\Big\}^{-1}, \end{aligned}$$

where $\alpha_1 = a_1c + c(x - 2dt)$, $\alpha_2 = b_1c - 2c^2t$. We plot the picture for this kind of solution by choosing the right parameters (Figs. 2(c) and 2(d)). This kind of solution behaves like the nonlinear superposition for the rogue wave solution and bright-dark soliton solution.

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