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Cylindrically Symmetric Solution in Teleparallel Theory

Gamal G. L. Nashed[1,2**]

[1]Centre for Theoretical Physics, The British University in Egypt, El-Sherouk City, Misr-Ismalia Desert Road, Postal No 11837, PO Box 43, Egypt

[2]Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt

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The field equations of a special class of teleparallel theory of gravitation and electromagnetic fields are applied to tetrad space having cylindrical symmetry with four unknown functions of radial coordinate r and azimuth angle θ. The vacuum stress-energy momentum tensor with one assumption concerning its specific form generates one non-trivial exact analytic solution. This solution is characterized by a constant magnetic field parameter B_0. If $B_0 = 0$, then the solution will reduce to the flat spacetime. The energy content is calculated using the superpotential given by Møller in the framework of teleparallel geometry. The energy contained in a sphere is found to be different from the pervious results.

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Theories of gravity based on the geometry of distance parallelism[1−7] are commonly considered as the closest alternative to the general relativity (GR) theory. Teleparallel gravity models possess a number of attractive features from both the geometrical and physical viewpoints. Teleparallelism is naturally formulated by gauging external (spacetime) translation and underlain the Weitzenböck spacetime[8] characterized by the metricity condition and by the vanishing of the curvature tensor. Translations are closely related to the group of general coordinate transformations which underlies GR. Therefore, the energy-momentum tensor represents the matter source in the field equations of tetradic theories of gravity like in GR.

Teleparallel theories of gravity, whose basic entities are the tetrad field $e_{a\mu}$ (a and μ are SO(3,1) and spacetime indices, respectively) have been considered a long time ago by Møller[9] in connection with attempts to define the energy of the gravitational field. Those theories of gravity have been considered for a long time in connection with attempts to define the energy of the gravitational field.[9,10] It is clear from the properties of the solutions of the Einstein field equation of an isolated system that a consistent expression for the energy density of the gravitational field would be given in terms of second order derivatives of the metric tensor. It is well known that there exists no covariant, nontrivial expression constructed out of the metric tensor, both in three and four dimensions that contain such derivatives. However, covariant expressions that contain second order derivatives of the tetrad fields are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum is related to the geometrical description of the gravitational field rather than being an intrinsic drawback of the theory.[11]

The problem of defining a consistent and unequivocal expression for the energy of the gravitational and electromagnetic fields is still open and an important question in GR. It is well know that the principle of equivalence has led to the belief that the gravitational energy cannot be localized.[12,13] However, the argument based on this principle regarding the nonlocalizibility of gravitational energy is controversial and not generally accepted.[11] Therefore, it is legitimate to conjecture that the difficulties associated to the problem of defining the gravitational energy is related to the geometrical description of the gravitational field, rather being an intrinsic nuisance of the theory.[11] The first attempts to define the energy of the gravitational field were based on pseudotensors,[14] which make use of coordinate dependent expressions. More recently, the idea of quasi-local energy, i.e., energy associated to a closed spacelike two surface, in the context of the Hilbert-Einstein action integral, has emerged as a tentative description of the gravitational energy.[16]

As stated above, the calculations of energy within the framework of GR theory have some problems.[9] It is the aim of the present work to study a tetrad having cylindrical symmetry and apply it to the field equations of gravitation and electromagnetic. Solving the resulting nonlinear differential equation we obtain an exact solution. We then calculate the energy using the superpotential of Mikhail *et al.*[16]

In teleparallel theory of gravity and electromagnetism the spacetime is represented by the Weitzenböck manifold W^4 of distance parallelism. This theory naturally arises within the gauge approach based on the group of the spacetime translations. Accordingly, at each point of this manifold, a gauge transformation is defined as a local translation of the tangent- space coordinate,[17] $x^a \to x'^a = x^a + b^a$ where $b^a = b^a(x^\mu)$ are the transformation parameters. For an infinitesimal transformation we have $\delta x^a = \delta b^c P_c x^a$ with δb^a being the infinitesimal parameters, and $P_a = \partial_a$ the generators of translations. Denoting the translational gauge potential by ${\Lambda^a}_\mu$, the gauge covariant derivative

**Email: nashed@bue.edu.eg

for a scalar field $\Phi(x^\mu)$ reads

$$D_\mu \Phi = e^a{}_\mu \partial_a \Phi, \quad \text{where} \quad e^a{}_\mu = \partial_\mu x^a + \Lambda^a{}_\mu, \tag{1}$$

is the tetrad field which satisfies the orthogonality condition $e^a{}_\mu e^\nu{}_a = \delta^\nu{}_\mu$. This nontrivial tetrad field induces a teleparallel structure on spacetime which is directly related to the presence of the gravitational field, and the Riemannian metric arises as

$$g^{\mu\nu} \stackrel{\text{def.}}{=} \eta^{ab} e_a{}^\mu e_b{}^\nu, \tag{2}$$

with the Minkowski metric $\eta^{ab} = \text{diag}(+1, -1, -1, -1)$. The gravitational Lagrangian L is an invariant constructed from $\gamma_{\mu\nu\rho}$ and $g^{\mu\nu}$, where $\gamma_{\mu\nu\rho}$ is the contorsion tensor given by

$$\gamma_{\mu\nu\rho} \stackrel{\text{def.}}{=} e^a{}_\mu e_{a\nu;\rho}, \tag{3}$$

where the semicolon denotes covariant differentiation with respect to Christoffel symbols. The most general Lagrangian density invariant under parity operation is given by the form[18]

$$\mathcal{L} \stackrel{\text{def.}}{=} (-g)^{1/2} \left(\alpha_1 \Phi^\mu \Phi_\mu + \alpha_2 \gamma^{\mu\nu\rho} \gamma_{\mu\nu\rho} + \alpha_3 \gamma^{\mu\nu\rho} \gamma_{\rho\nu\mu} \right), \quad \text{where} \quad g \stackrel{\text{def.}}{=} \det(g_{\mu\nu}), \tag{4}$$

and Φ_μ is the basic vector field defined by

$$\Phi_\mu \stackrel{\text{def.}}{=} \gamma^\rho{}_{\mu\rho}. \tag{5}$$

Here α_1, α_2, and α_3 are constants determined by Møller such that the theory coincides with GR in the weak fields:[18]

$$\alpha_1 = -\frac{1}{\kappa}, \qquad \alpha_2 = \frac{\lambda}{\kappa}, \qquad \alpha_3 = \frac{1}{\kappa}(1 - \lambda), \tag{6}$$

where κ is the Einstein constant and λ is a free dimensionless parameter (throughout this study we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$). The same choice of the parameters was also obtained by Hayashi and Nakano.[19]

The electromagnetic Lagrangian density $L_{e.m.}$ is given by[20]

$$L_{\text{e.m.}} = -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{7}$$

where $F_{\mu\nu}$ is given by (Heaviside Lorentz rationalized unites will be used throughout this study)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{8}$$

and A_μ is the electromagnetic potential.

The gravitational and electromagnetic field equations for the system described by $L_G + L_{\text{e.m.}}$ are as follows:

$$G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu}, \; K_{\mu\nu} = 0, \; \partial_\nu \left(\sqrt{-g} F^{\mu\nu} \right) = 0, \tag{9}$$

where the Einstein tensor $G_{\mu\nu}$ is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{10}$$

Here $H_{\mu\nu}$ and $K_{\mu\nu}$ are given by

$$H_{\mu\nu} \stackrel{\text{def.}}{=} \lambda \Big[\gamma_{\rho\sigma\mu} \gamma^{\rho\sigma}{}_\nu + \gamma_{\rho\sigma\mu} \gamma_\nu{}^{\rho\sigma} + \gamma_{\rho\sigma\nu} \gamma_\mu{}^{\rho\sigma} + g_{\mu\nu} \Big(\gamma_{\rho\sigma\lambda} \gamma^{\lambda\sigma\rho} - \frac{1}{2} \gamma_{\rho\sigma\lambda} \gamma^{\rho\sigma\lambda} \Big) \Big], \tag{11}$$

and

$$K_{\mu\nu} \stackrel{\text{def.}}{=} \lambda \big[\Phi_{\mu,\nu} - \Phi_{\nu,\mu} - \Phi_\rho \big(\gamma^\rho{}_{\mu\nu} - \gamma^\rho{}_{\nu\mu} \big) + \gamma_{\mu\nu}{}^\rho{}_{;\rho} \big], \tag{12}$$

and they are symmetric and skew symmetric tensors, respectively. The energy-momentum tensor $T^{\mu\nu}$ is given by

$$T^{\mu\nu} = g_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} - \frac{1}{4} g^{\mu\nu} g^{\lambda\rho} g^{\epsilon\sigma} F_{\lambda\epsilon} F_{\rho\sigma}. \tag{13}$$

The tetrad space having cylindrical symmetry is given by

$$(e_i{}^\mu) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B \sin\theta \cos\phi & \frac{B_1}{r} \cos\theta \cos\phi & -\frac{B_2 \sin\phi}{r \sin\theta} \\ 0 & B \sin\theta \sin\phi & \frac{B_1}{r} \cos\theta \sin\phi & \frac{B_2 \cos\phi}{r \sin\theta} \\ 0 & B \cos\theta & -\frac{B_1}{r} \sin\theta & 0 \end{pmatrix} \tag{14}$$

where A, B, B_1 and B_2 are unknown functions in r and θ.

Applying Eq. (14) to the field Eqs. (9) we note that the two tensors $H_{\mu\nu}$ and $K_{\mu\nu}$ are vanishing identically regardless to the value of the functions A, B, B_1 and B_2. Thus Møller field equations reduce for the tetrad Eq. (14) to Einstein's field equations. Applying the transformation

$$R = \frac{r}{B}, \tag{15}$$

to tetrad Eq. (14) we obtain

$$(e_i{}^\mu) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \mathcal{X} \sin\theta \cos\phi & \frac{B_1}{rB} \cos\theta \cos\phi & -\frac{B_2 \sin\phi}{rB \sin\theta} \\ 0 & \mathcal{X} \sin\theta \sin\phi & \frac{B_1}{rB} \cos\theta \sin\phi & \frac{B_2 \cos\phi}{rB \sin\theta} \\ 0 & \mathcal{X} \cos\theta & -\frac{B_1}{rB} \sin\theta & 0 \end{pmatrix}, \quad \mathcal{X} = (1 - rB') \tag{16}$$

with $B' = dB(r)/dr$. Then the field equations (9) take an intricate form. Here we will write the non-vanishing components of Einstein tensor $G_{\mu\nu}$ related to Eq. (9) that have the form

$$\begin{aligned} -\kappa T_{00} &= G_{00}, \quad -\kappa T_{11} = G_{11}, \\ -\kappa T_{12} &= G_{12}, \quad -\kappa T_{21} = G_{21}, \\ -\kappa T_{22} &= G_{22}, \quad -\kappa T_{33} = G_{33}, \end{aligned} \tag{17}$$

where $G_{\mu\nu}$ is defined by Eq. (10).

Now we are going to find a special solution to the nonlinear partial differential equations (17). When

$$\begin{aligned} A &= \frac{1}{\Lambda}, \quad B = \frac{1}{2} \ln(4\sqrt{\Lambda}), \\ B_1 &= \frac{B}{\Lambda}, \quad B_2 = B\Lambda, \end{aligned} \tag{18}$$

where Λ is defined as

$$\Lambda = 1 + \frac{1}{4}{B_0}^2 r^2 \sin^2\theta. \tag{19}$$

The form of the vector potential A_μ, the antisymmetric electromagnetic tensor field $F_{\mu\nu}$ and the stress energy-momentum tensor ${T_\mu}^\nu$ are given by

$$\begin{aligned}
A_t &= B_0 r\cos\theta,\\
\boldsymbol{F} &= B_0\left(\cos\theta dt\wedge dr + r\sin\theta d\theta\wedge dt\right),\\
{T_0}^0 &= -{T_3}^3 = \frac{{B_0}^2}{8\pi\Lambda^4},\\
{T_1}^1 &= -{T_2}^2 = \frac{{B_0}^2(1-2\sin^2\theta)}{8\pi\Lambda^4},\\
{T_1}^2 &= r^2{T_2}^1 = \frac{-{B_0}^2\sin\theta\cos\theta}{8\pi r\Lambda^4}.
\end{aligned} \tag{20}$$

Using Eq. (18) in Eq. (16) we obtain

$$({e_i}^\mu) = \begin{pmatrix} \Lambda & 0 & 0 & 0\\ 0 & \frac{\sin\theta\cos\phi}{\Lambda} & \frac{\cos\theta\cos\phi}{r\Lambda} & -\frac{\Lambda\sin\phi}{r\sin\theta}\\ 0 & \frac{\sin\theta\sin\phi}{\Lambda} & \frac{\cos\theta\sin\phi}{r\Lambda} & \frac{\Lambda\cos\phi}{r\sin\theta}\\ 0 & \frac{\cos\theta}{\Lambda} & -\frac{\sin\theta}{r\Lambda} & 0 \end{pmatrix}, \tag{21}$$

where Λ is given by Eq. (19). The associated Riemannian metric of tetrad Eq. (21) takes the form

$$ds^2 = \Lambda^2\left(dt^2 - dr^2 - r^2 d\theta^2\right) - \frac{r^2\sin^2\theta}{\Lambda^2}d\phi^2, \tag{22}$$

which is the line-element given before by Melvin.[21]

In what follows we will calculate the energy using the superpotential derived form Møller's theory by Mikhail *et al.*[16] which has the form

$$\begin{aligned}
{\mathcal{U}_\mu}^{\nu\lambda} = \frac{(-g)^{1/2}}{2\kappa}{P_{\chi\rho\sigma}}^{\tau\nu\lambda}\big[&\Phi^\rho g^{\sigma\chi}g_{\mu\tau} - \lambda g_{\tau\mu}\gamma^{\chi\rho\sigma}\\
&-(1-2\lambda)g_{\tau\mu}\gamma^{\sigma\rho\chi}\big],
\end{aligned} \tag{23}$$

where ${\mathcal{U}_\mu}^{\nu\lambda}$ is the superpotential derived from Møller's theory and ${P_{\chi\rho\sigma}}^{\tau\nu\lambda}$ is defined as

$${P_{\chi\rho\sigma}}^{\tau\nu\lambda} \stackrel{\text{def.}}{=} {\delta_\chi}^\tau {g_{\rho\sigma}}^{\nu\lambda} + {\delta_\rho}^\tau {g_{\sigma\chi}}^{\nu\lambda} - {\delta_\sigma}^\tau {g_{\chi\rho}}^{\nu\lambda}, \tag{24}$$

with ${g_{\rho\sigma}}^{\nu\lambda}$ being a tensor defined by

$${g_{\rho\sigma}}^{\nu\lambda} \stackrel{\text{def.}}{=} {\delta_\rho}^\nu{\delta_\sigma}^\lambda - {\delta_\sigma}^\nu{\delta_\rho}^\lambda. \tag{25}$$

The energy is expressed by the surface integral[22]

$$E = \lim_{r\to\infty}\int_{r=\text{constant}} {\mathcal{U}_0}^{0\alpha} n_\alpha ds, \tag{26}$$

where n_α is the unit three-vector normal to the surface element dS.

Now we are in a position to calculate the energy associated with solution (18) using superpotential (23). As is clear from Eq. (26), the only component which contributes to the energy is ${\mathcal{U}_0}^{0\alpha}$. Thus substituting Eq. (18) into Eq. (23) we obtain the non-vanishing value,

$$\begin{aligned}
{\mathcal{U}_0}^{01} &= \frac{{B_0}^2 x\left[{B_0}^2(x^2+y^2)+8\right]}{128\pi},\\
{\mathcal{U}_0}^{02} &= \frac{{B_0}^2 y\left[{B_0}^2(x^2+y^2)+8\right]}{128\pi},\\
{\mathcal{U}_0}^{03} &= 0.
\end{aligned} \tag{27}$$

Substituting Eq. (27) into Eq. (26) we obtain

$$E(r) = \frac{1}{6}{B_0}^2 r^3 + \frac{1}{60}{B_0}^4 r^5. \tag{28}$$

This result depends on the constant B_0, as is expected.

We have studied a cylindrically symmetric solution in the teleparallel theory of gravitation and electromagnetic fields. The axial vector part of the torsion, a^μ of this solution is identically vanishing. The associated metric of Eq. (21) gives rise to the same Riemannian metric given before by Melvin.[21,23] Energy distribution in Melvin's universe have been computed by many authors[12,25] using different definitions of the energy momentum complexes within the framework of GR theory.

It was shown by Møller[26] that a tetrad description of a gravitational field equation allows a more satisfactory treatment of the energy-momentum complex than GR. Therefore, we have applied the superpotential method given by Mikhail *et al.*[16] to calculate the energy. As is clear from Eq. (28) that energy depends on the constant magnetic B_0 and if this constant equal to zero, then $E = 0$ since solution (18) will reduces to a flat spacetime, i.e. $ds^2 = dt^2 - dr^2 - r^2(d\theta^2 - \sin\theta^2 d\phi^2)$.

Table 1. Comparison between the results of energy using different energy momentum complexes.

Energy momentum complex	Energy	Equation
Møller Within the Framework of Tetrad Space-time	$\frac{1}{6}{B_0}^2r^3 + \frac{1}{60}{B_0}^4r^5$	(28)
Møller Within the Framework of GR Space-time	$\frac{1}{3}{B_0}^2r^3 - \frac{1}{15}{B_0}^4r^5 + \frac{1}{70}{B_0}^6r^7$	(29)
Landau and Lifshitz, Papapetrou Within the Framework of GR Space-time	$\frac{1}{6}{B_0}^2r^3 + \frac{1}{20}{B_0}^4r^5 + \frac{1}{140}{B_0}^6r^7 + \frac{1}{2520}{B_0}^8r^9$	(30)

It is of interest to mention here that Radinschi[24] has calculated the energy distribution of Melvin metric using Møller energy momentum complex within the framework of GR and has obtained

$$E(r) = \frac{1}{3}{B_0}^2 r^3 - \frac{1}{15}{B_0}^4 r^5 + \frac{1}{70}{B_0}^6 r^7. \tag{29}$$

Also Xulu[25] has calculated the energy of the same metric using the energy momentum complexes given by Landau and Lifshitz and by Papapetrou and he has obtained

$$E(r) = \frac{1}{6}{B_0}^2 r^3 + \frac{1}{20}{B_0}^4 r^5 + \frac{1}{140}{B_0}^6 r^7 + \frac{1}{2520}{B_0}^8 r^9. \tag{30}$$

It is clear that there is a difference between the results obtained by Radinschi, Xulu and the results obtained in this study. The difference of energy of the same spacetime within GR is due to the fact that there are many definitions of energy momentum complexes.[14] However, this problem does not exist in the teleparallel theories because it has only one definition of the energy-momentum complex.[18]

A comparison between the results of the energy using different energy momentum complexes are given in Table 1.

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