Approximate Solution of Homotopic Mapping to Solitary Wave for Generalized Nonlinear KdV System *

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(Received 1 September 2008)

We study a generalized nonlinear KdV system is studied by using the homotopic mapping method. Firstly, a homotopic mapping transform is constructed; secondly, the suitable initial approximation is selected; then the homotopic mapping is used. The accuracy of the approximate solution for the solitary wave is obtained. From the obtained approximate solution, the homotopic mapping method exhibits a good accuracy.

PACS: 02.30.Jr, 02.30.Mv, 02.60.Lj, 02.30.Sa

Solitons and chaos are two most important notions of nonlinear science, which have been widely applied in natural sciences such as chemistry, biology, mathematics, communication, and particularly in almost all branches of physics such as fluid dynamics, plasma, field theory, optics, condensed matter.[1−6] Recently, more new methods are appeared, for example, method of hyperbolic tangent function, homogeneous equilibrium method, method of Jacobi elliptic function, method of auxiliary equation, etc.[7,8] Many scholars have carried out a great deal of work, for example, shock wave, scattering light wave, quantum mechanics, atmospheric physics, network of neurons and so on have been studied for the theorem of solitary wave.[9−11] The asymptotic method for nonlinear theory of solitary wave is a new study. The main essential of this method is that studied nonlinear problems deal with linear problems by using the asymptotic expansion. The homotopic mapping method[12,13] is such a new method.

In the past decade, many approximate methods for the nonlinear problem have been developed and refined, including the method of averaging, boundary layer method, methods of matched asymptotic expansion and multiple scales. Recently, many scholars such as Ni and Wei,[14] Bartier,[15] Llibre and de Silva[16] and Guarguaglini and Natalini[17] have carried out a great deal of work. Using the method of differential inequalities and others, researchers also considered a class of reaction diffusion problems,[18] activator inhibitor systems,[19] ecological environment,[20] shock wave,[21] soliton wave,[22,23] laser pulse,[24] ocean science,[25,26] atmospheric physics,[27−30] etc. In this Letter, we consider a class of generalized nonlinear KdV systems, and obtain approximate solution of solitary wave.

Consider the following generalized nonlinear KdV system:[31]

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x\partial y^2}-2au\frac{\partial v}{\partial y}-av\frac{\partial u}{\partial y}=f(u,v), \quad (1)$$

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0, \quad (2)$$

where a is a constant, $f(u,v)$ is a sufficiently smooth function with its arguments.

From the KdV system (1)–(2) as $f(u,v)=0$ and letting $u=w_y$, $v=w_x$, we have

$$\frac{\partial^2 w}{\partial t\partial y}-\frac{\partial^4 w}{\partial x\partial y^3}-2a\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=0. \quad (3)$$

Now we first search the solitary wave solution for Eq. (3) by using the method in Ref. [31].

Assume that the solution w of Eq. (3) reads

$$w=A(t,y)+B(t,x,y)\phi(z), \quad (4)$$

$$\phi'=\phi^2, \quad (5)$$

where A,B,z and ϕ are to be determined functions and the prime denotes the first derivative with respect to z.

From Eqs. (4) and (5), we have

$$w_x=B_x\phi+Bz_x\phi^2, \quad w_y=A_y+B_y\phi+Bz_y\phi^2,$$
$$w_t=A_t+B_t\phi+Bz_t\phi^2, \quad (6)$$

$$w_{xy}=B_{xy}\phi+(B_xz_y+(Bz_x)_y)\phi^2+2Bz_xz_y\phi^3, \quad (7)$$

$$w_{yy}=B_{yy}\phi+(B_yz_y+(Bz_y)_y)\phi^2+2Bz_y^2\phi^3, \quad (8)$$

$$w_{ty}=A_{ty}+B_{ty}\phi+(B_tz_y+(Bz_t)_y)\phi^2+2Bz_tz_y\phi^3, \quad (9)$$

*Supported by the National Natural Science Foundation of China under Grant No 40676016, the National Basic Research Programme of China under Grant Nos 2003CB415101-03 and 2004CB418304, the Knowledge Innovation Project of Chinese Academy of Sciences under Grant No KZCX3-SW-221, in part by E-Institutes of Shanghai Municipal Education Commission under Grant No E03004.

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$$\begin{aligned} w_{xyyy} =& B_{xyyy}\phi + (B_{xyy}z_y + (B_{yy}z_x)_y \\ &+ (B_y z_y + (Bz_y)_y)_{xy})\phi^2 \\ &+ (2B_{yy}z_x z_y + 2((B_y z_y + (Bz_y)_y)_x z_y \\ &+ 2(B_y z_x z_y + (Bz_y)_y z_x)_y + 2(Bz_y^2)_{xy})\phi^3 \\ &+ 6(B_y z_x z_y^2 + (Bz_y)_y z_x z_y + (Bz_y^2)_x z_y \\ &+ (Bz_y^2 z_x)_y)\phi^4 + 24Bz_y^3 z_x \phi^5. \end{aligned} \quad (10)$$

To determine w explicitly, we can substitute Eqs. (6)–(10) into Eq. (3) and collect coefficients of polynomials of ϕ and set each coefficient to be zero, then we can obtain

$$\begin{aligned} A =& -\frac{1}{2a}\int \frac{1}{z_x z_y^2}\big[-2z_{xyy}z_y^2 - 2z_x z_y z_{yyy} \\ &+ 2z_{xy}z_y z_{yy} + z_x z_{yy}^2 + z_t z_y^2\big]\, dx, \end{aligned} \quad (11)$$

$$B = -4z_y/a, \quad (12)$$

$$\phi = -1/z. \quad (13)$$

$$z = F(y) + G(x - ct), \quad (14)$$

where F and G are two arbitrary separation functions y and $x - ct$ with constant c, respectively. From Eqs. (4), (11)–(14) and $u = w_y$, $v = w_x$, it is easy to see that we can derive the following soliton wave of system (1)–(2) as $f(u, v) = 0$:

$$\begin{aligned} \bar{u}(t, x, y) =& \frac{F_{yy}^2(y) - cF_y^2(y) - 2F_y(y)F_{yyy}(y)}{2aF_y^2(y)} \\ &+ \frac{4F_{yy}(y)}{a(F(y) + G(x - ct))} \\ &- \frac{4F_y^2(y)}{a(F(y) + G(x - ct))^2}, \end{aligned} \quad (15)$$

$$\bar{v}(t, x, y) = \frac{-4F_y(y)G_x(x - ct)}{a(F(y) + G(x - ct))^2}. \quad (16)$$

Now we consider the generalized nonlinear KdV system (1)–(2). Introducing a homotopic mapping $H_i(u, v, p):\ R^2 \times I \to R\ (i = 1, 2)$:

$$\begin{aligned} H_1(u, v, p) =& L_1[u] - L_1[\bar{u}] + p(L_1[\bar{u}] - 2auv_y \\ &- avu_y - f(u, v)), \end{aligned} \quad (17)$$

$$H_2(u, v, p) = L_2[u, v] + (p - 1)L_2[\bar{u}, \bar{v}], \quad (18)$$

where $R = (-\infty, +\infty)$, $I = [0, 1]$, $(\bar{u}, \bar{v})$ is a set of initial approximation of system (1)–(2) and the linear operators $L_i (i = 1, 2)$ are

$$L_1[u] = \frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial x \partial y^2}, \quad L_2[u, v] = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}.$$

We now select initial approximation $(\bar{u}, \bar{v})$ of system (1)–(2), which satisfies

$$L_1[\bar{u}] = 2a\bar{u}\frac{\partial \bar{v}}{\partial y} + a\bar{v}\frac{\partial \bar{u}}{\partial y}, \quad (19)$$

$$L_2[\bar{u}, \bar{v}] = 0. \quad (20)$$

From the above discussion, a set of solution $(\bar{u}, \bar{v})$ of the system (19)–(20) can be expressed by Eqs. (15) and (16).

Obviously, from mapping (17) and (18), $H_i(u, v, 1) = 0$ $(i = 1, 2)$ are the same as Eqs. (1) and (2). Thus the solution (u, v) of the system (1)–(2) is the same as the solution of $H_i(u, v, p) = 0$ $(i = 1, 2)$ as $p \to 1$.

Let

$$u = \sum_{i=0}^{\infty} u_i(t, x, y)p^i, \quad v = \sum_{i=0}^{\infty} v_i(t, x, y)p^i. \quad (21)$$

Substituting Eq. (21) into Eqs. (17) and (18), comparing the coefficients of Eq. $H_i(u, v, p) = 0$ $(i = 1, 2)$ for the same powers in p, from the coefficients zeroth order power in p for $H_i(u, v, p) = 0$ we have

$$L_1[u_0] = L_1[\bar{u}], \quad L_2[u_0, v_0] = L_2[\bar{u}, \bar{v}]. \quad (22)$$

We now select a set of (u_0, v_0) to be $(\bar{u}_0, \bar{v}_0)$. Then from Eqs. (19), (20), (15), and (16), we can obtain

$$\begin{aligned} u_0(t, x, y) =& \frac{F_{yy}^2(y) - cF_y^2(y) - 2F_y(y)F_{yyy}(y)}{2aF_y^2(y)} \\ &+ \frac{4F_{yy}(y)}{a(F(y) + G(x - ct))} \\ &- \frac{4F_y^2(y)}{a(F(y) + G(x - ct))^2}, \end{aligned} \quad (23)$$

$$v_0(t, x, y) = \frac{-4F_y(y)G_x(x - ct)}{a(F(y) + G(x - ct))^2}. \quad (24)$$

From the coefficient first order power in p for $H_i(u, v, p) = 0$ $(i = 1, 2)$, we have

$$L_1[u_1] = f(u_0, v_0), \quad (25)$$

$$L_2[u_1, v_1] = 0, \quad (26)$$

where u_0 and v_0 are represented by Eqs. (23) and (24), respectively. Using the Fourier transform method, a set of solution of system (25)–(26) with zero initial value will read

$$\begin{aligned} u_1(t, x, y) =& \frac{1}{4\pi^2}\int_0^t dt_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\big(u_0(\tau, \xi, \eta), \\ & v_0(\tau, \xi, \eta)\big) \cdot F_1(t, x, y; \tau, \xi, \eta)d\xi d\eta d\tau, \end{aligned} \quad (27)$$

$$\begin{aligned} v_1(t, x, y) =& \frac{1}{4\pi^2}\int_0^t dt_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{y} \big[f\big(u_0(\tau, \xi, \eta), \\ & v_0(\upsilon, \xi, \eta)\big) \cdot F_1(t_1, x, y_1 \tau, \xi, \eta)\big]_\xi \\ & \cdot dy_1 d\xi d\eta d\tau, \end{aligned} \quad (28)$$

where

$$\begin{aligned} F_1(t, x, y, ; \tau, \xi, \eta) =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\big(-i(\lambda_1\lambda_2^2(t - \tau) \\ & - \lambda_1(x - \xi) - \lambda_2(y - \eta))\big)d\lambda_1 d\lambda_2. \end{aligned}$$

From the coefficient second order power in p for $H_i(u,v,p)=0$ $(i=1,2)$, we have

$$L_1[u_2]=f_u(u_0,v_9)u_1+f_v(u_0,v_9)v_1-2a(u_1v_{0y}+u_0\bar{v}_{1y})-a(v_1u_{0y}+v_0u_{1y})\equiv\bar{f}(t,x,y),\tag{29}$$

$$L_2[u_2,v_2]=0.\tag{30}$$

Thus, a set of solution of system (29)–(30) with zero initial value will read

$$u_2(t,x,y)=\frac{1}{4\pi^2}\int_0^t dt_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{f}(\tau,\xi,\eta)\cdot F_1(t,x,y;\tau,\xi,\eta)d\xi d\eta d\tau,\tag{31}$$

$$v_1(t,x,y)=\frac{1}{4\pi^2}\int_0^t dt_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{y}\bar{f}(\tau,\xi,\eta)\cdot F_1(t,x,y_1;\tau,\xi,\eta)]_\xi dy_1d\xi d\eta d\tau.\tag{32}$$

From Eqs. (23), (24), (27), (28), (31), (32) and the homotopic theory, we obtain the second order approximation of solitary wave solution for the generalized nonlinear KdV system (1)–(2):

$$u(t,x,y)=\frac{F_{yy}^2(y)-cF_y^2(y)-2F_y(y)F_{yyy}(y)}{2aF_y^2(y)}+\frac{4F_{yy}(y)}{a(F(y)+G(x-ct))}-\frac{4F_y^2(y)}{a(F(y)+G(x-ct))^2}+\frac{1}{4\pi^2}\int_0^t dt_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big[f\big(u_0(\tau,\xi,\eta),v_0(\tau,\xi,\eta)\big)+\bar{f}(\tau,\xi,\eta)\big]\times F_1(t,x,y;\tau,\xi,\eta)d\xi d\eta d\tau,\tag{33}$$

$$v(t,x,y)=\frac{-4F_y(y)G_x(x-ct)}{a(F(y)+G(x-ct))^2}+\frac{1}{4\pi^2}\int_0^t dt_1\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{y}\big[[f(u_0(\tau,\xi,\eta),v_0(\tau,\xi,\eta))+\bar{f}(\tau,\xi,\eta)]\times F_1(t_1,x,y_1;\tau,\xi,\eta)\big]_\xi dy_1d\xi d\eta d\tau.\tag{34}$$

Consequently, using the same method, we can also obtain the higher order approximation of solitary wave solution for the generalized nonlinear KdV system (1)–(2).

In order to explain the accuracy of the expressions of approximate solutions (33) and (34), now we consider the small perturbation term $f(u,v)=\varepsilon\sin(u+v)$, $0<\varepsilon\ll 1$ in Eq. (1).

Firstly, from Eqs. (33) and (34), we can obtain a set of approximate solution (u,v) by using the homotopic mapping method as follows:

$$u(t,x,y)=\frac{F_{yy}^2(y)-cF_y^2(y)-2F_y(y)F_{yyy}(y)}{2aF_y^2(y)}+\frac{4F_{yy}(y)}{a(F(y)+G(x-ct))}-\frac{4F_y^2(y)}{a(F(y)+G(x-ct))^2}+\frac{\varepsilon}{4\pi^2}\int_0^t dt_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\big[(\sin(u_0+v_0)-2a(\bar{u}_1v_{0y}+u_0\bar{v}_{1y})-a(\bar{v}_1u_{0y}+v_0\bar{u}_{1y})\big]\times F_1(t,x,y;\tau,\xi,\eta)d\xi d\eta d\tau+O(\varepsilon^2),\quad 0<\varepsilon\ll 1,\tag{35}$$

$$v(t,x,y)=\frac{-4F_y(y)G_x(x-ct)}{a(F(y)+G(x-ct))^2}+\frac{\varepsilon}{4\pi^2}\int_0^t dt_1\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{y}\big[[\sin(u_0+v_0)-2a(\bar{u}_1v_{0y})-a(\bar{v}_1u_{0y}+\bar{v}_0u_{1y})]\times F_1(t_1,x,y_1;\tau,\xi,\eta)\big]_\xi dy_1d\xi d\eta d\tau+O(\varepsilon^2),\quad 0<\varepsilon\ll 1,\tag{36}$$

where u_0, v_0 and $u_1=\varepsilon\bar{u}_1$, $v_1=\varepsilon\bar{v}_1$ denote by Eqs. (23), (24) and (27), (28) as $f(u,v)=\varepsilon\sin(u+v)$, respectively.

On the other hand, we construct the asymptotic solution $(\tilde{u},\tilde{v})$ of system (1)–(2) as $f(u,v)=\varepsilon\sin(u+v)$ by using the perturbation method. Let

$$\tilde{u}=\sum_{i=0}^{\infty}\tilde{u}_i(t,x,y)\varepsilon^i,\quad \tilde{u}=\sum_{i=0}^{\infty}\tilde{v}_i(t,x,y)\varepsilon^i,\quad 0<\varepsilon\ll 1.\tag{37}$$

Substituting Eq. (37) into system (1)–(2) as $f(u,v)=\varepsilon\sin(u+v)$ in ε, comparing the coefficients for the same powers in ε, we can obtain $(\tilde{u}_0,\tilde{v}_0)$, $(\tilde{u}_1,\ \tilde{v}_1)$. From Eq. (37), we have

$$\tilde{u}(t,x,y)=\frac{F_{yy}^2(y)-cF_y^2(y)-2F_y(y)F_{yyy}(y)}{2aF_y^2(y)}+\frac{4F_{yy}(y)}{a(F(y)+G(x-ct))}-\frac{4F_y^2(y)}{a(F(y)+G(x-ct))^2}+\frac{\varepsilon}{4\pi^2}\int_0^t dt_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[(\sin(u_0+v_0)-2a(\bar{u}_1v_{0y}+u_0\bar{v}_{1y})-a(\bar{v}_1u_{0y}+v_0\bar{u}_{1y})]\times F_1(t,x,y;\tau,\xi,\eta)d\xi d\eta d\tau+O(\varepsilon^2),\quad 0<\varepsilon\ll 1,\tag{38}$$

$$\tilde{v}(t,x,y)=\frac{-4F_y(y)G_x(x-ct)}{a(F(y)+G(x-ct))^2}+\frac{\varepsilon}{4\pi^2}\int_0^t dt_1\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{y}\big[[\sin(u_0+v_0)-2a(\bar{u}_1v_{0y})-a(\bar{v}_1u_{0y}+\bar{v}_0u_{1y})]\times F_1(t_1,x,y_1;\tau,\xi,\eta)\big]_\xi\cdot dy_1d\xi d\eta d\tau+O(\varepsilon^2),\quad 0<\varepsilon\ll 1.\tag{39}$$

Comparing Eqs. (35), (36) and (38), (39) respectively, we find that they are the same. Therefore, it may be found that the approximate solution of solitary wave for the generalized nonlinear KdV system (1)–(2) by using the homotopic mapping method possesses a good accuracy.

Solitary wave denotes a class of complicated natural phenomena. Hence we solve them using the approximate method. The homotopic mapping method is a simple and valid method.

The homotopic mapping method is an approximate analytic method, which differs from general numerical method. The expansions of solution using the homotopic mapping method can be continuously analytical operation. Thus, from Eqs. (34) and (35), we can study further the qualitative and quantitative behaviour of solitary wave elsewhere.

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