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Nonlinear Landau–Zener Tunnelling with Two and Three-Body Interactions *

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We investigate the nonlinear Landau–Zener tunnelling of Bose–Einstein condensate (BEC) in an accelerating optical lattice with two- and three-body interactions between the particles. The influence of the three-body interaction on the eigenstates and the transition probability are discussed both analytically and numerically. The analytical eigenstates and the tunnelling probability are obtained, which are verified by numerical methods. It is shown that the eigenstates and the tunnelling probability are modified dramatically by three-body interaction.

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Bose–Einstein condensates (BEC) in optical lattices has been attracted increasing interest. The related phenomena of BEC in optical lattices, such as Bloch oscillation,[1−4] Landau–Zener tunnelling,[5−8] energetic and dynamical instability,[9−11] self-trapping phenomenon,[12−15] have been investigated. One of the important discovery is that the energy band structure and the tunnelling properties are influenced dramatically by the interatomic interactions. It is shown that, because of the repulsive interatomic interactions, a loop structure will appear at the boundary of the first Brillouin zone in the lowest Bloch band.[5] However, theoretical studies of BEC in optical lattices are mainly focused on considering the two-body interactions. It is clear that in low temperature and density, where interatomic distances are much greater than the distance scale of interatomic interactions, two-body s-wave scattering should be important and three-body interactions can be neglected. On the other hand, if the atom density is high, for example, in the case of the miniaturization of the devices in the integrated atom optics, three-body interactions can start to play an important role.[18−23] As reported in Ref. [24], even for a small strength of the three-body interactions, the region of stability for the condensate can be extended considerably. The energy band structure and the stability of BEC are also influenced dramatically by the three-body interactions.[16] In this Letter, we focus on studying the nonlinear tunnelling properties and the nonlinear eigenstates of BEC with two- and three-body interactions loaded into an one-dimensional accelerating optical lattice. By using the two-mode approximation, the nonlinear eigenstates and the nonlinear tunnelling properties are discussed both analytically and numerically. The results show that the three-body interaction modifies the eigenstates and the tunnelling probability dramatically.

We focus on the situation that a BEC with two- and three-body interactions is loaded into an one-dimensional accelerating optical lattice. Due to the tight confinement perpendicular to the optical lattice from the waveguide potential, the dynamics of this system is largely one-dimensional. Integrating out the two perpendicular directions,[16] the dynamics of the BEC can be modelled by the 1D-GP equation in the comoving frame of the lattice,[5,8,16,25]

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{1}{2m}\Big(\hbar\frac{\partial}{\partial t} - ima_l t\Big)^2\Phi + v_0\cos(2k_L x)\Phi + \frac{2\hbar^2 a_s}{a_\perp^2 m}|\Phi|^2\Phi + \frac{g_2}{3\pi^2 a_\perp^4}|\Phi|^4\Phi, \quad (1)$$

where $\Phi(x,t)$ is the wavefunction of the condensate, m is the mass of the atoms, a_s is the two-body s-wave scattering length, v_o is the strength of the periodic potential, k_L is the wave number of the laser light which is used to generate the optical lattice. A force of ma_l is represented in the vector potential gauge, which may stand for either the inertial force in the comoving frame of an accelerating lattice or the gravity force, $a_\perp = \sqrt{\dfrac{\hbar}{m\omega_\perp}}$, $\omega_\perp$ is the radial frequencies of the anisotropic harmonic trap. The term $\dfrac{g_2}{3\pi^2 a_\perp^4}|\Phi|^4\Phi$ is a three-body interaction correction to the G-P equation.

For convenience, we cast Eq. (1) into the dimensionless form

$$i\frac{\partial\Phi}{\partial t} = -\frac{1}{2}\Big(\frac{\partial}{\partial t} - i\alpha t\Big)^2\Phi + v\cos(x)\Phi + c|\Phi|^2\Phi + \lambda|\Phi|^4\Phi, \quad (2)$$

where the variables are scaled as

$$x \sim 2k_L x, \quad \Phi \sim \frac{\Phi}{\sqrt{2k_L N}}, \quad t \sim \frac{4\hbar}{m}k_L^2 t. \quad (3)$$

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The dimensionless variables in Eq. (2) are defined as

$$v=\frac{mv_0}{4\hbar^2k_L^2},\quad \alpha=\frac{m^2}{8\hbar^2k_L^3}a_l,$$
$$c=\frac{Na_s}{k_La_\perp^2},\quad \lambda=\frac{mg_2N^2}{3\pi^2\hbar^2a_\perp^4}, \tag{4}$$

where N is the total numbers of atoms, c and λ represent the effective two- and three-body interactions. Recent theoretical studies estimate[21−23] that for ^{87}Rb atoms, $g_2\approx\hbar\times10^{-26}\,\mathrm{cm}^6/\mathrm{s}\sim\hbar\times10^{-27}\,\mathrm{cm}^6/\mathrm{s}$. Taking $N\approx10^4-10^5$, $a_\perp\approx10^{-6}\,\mathrm{m}$, $k_L=\dfrac{2\pi}{976nm}$, the three-body interaction is expected to be positive with a value of $0<\lambda<1$.

In the neighbourhood of the Brillouin zone edge $k=1/2$, the wavefunction can be approximated by[5]

$$\Phi(x,t)=a(t)e^{ikx}+b(t)e^{i(k-1)x}, \tag{5}$$

where $|a|^2+|b|^2=1$. Substituting this into Eq. (2) and comparing the coefficients of e^{ikx} and $e^{i(k-1)x}$, we have

$$i\frac{\partial a}{\partial t}=\frac{1}{2}(k-\alpha t)^2a+\frac{v}{2}b+c(1+|b|^2)a+\lambda(1+2|a|^2|b|^2+2|b|^2)a, \tag{6}$$
$$i\frac{\partial b}{\partial t}=\frac{1}{2}(k-1-\alpha t)^2b+\frac{v}{2}a+c(1+|a|^2)b+\lambda(1+2|a|^2|b|^2+2|a|^2)b, \tag{7}$$

Eqs. (6) and (7) are the two-level models which describe the tunnelling dynamics of BEC with both two- and three-body interactions in optical lattice. Here, we focus on discussing the tunnelling properties at the Brillouin zone edge, i.e., $k=1/2$, so after the linearization of the quadratic kinetic terms around $k=1/2$ and dropping a constant energy of $c[1+(|a|^2+|b|^2)/2]$, the Hamiltonian of Eqs. (6) and (7) is

$$\boldsymbol{H}=\begin{pmatrix}\mathcal{A} & v/2\\ v/2 & \mathcal{B}\end{pmatrix}, \tag{8}$$
$$\mathcal{A}=\frac{\gamma}{2}+\frac{c}{2}(|b|^2-|a|^2)+\lambda(1+2|a|^2|b|^2+2|b|^2),$$
$$\mathcal{B}=-\frac{\gamma}{2}-\frac{c}{2}(|b|^2-|a|^2)+\lambda(1+2|a|^2|b|^2+2|a|^2).$$

The Hamiltonian is characterized by four parameters: the linear coupling constant v between the two levels, the level bias $\gamma(t)=\alpha t$, where α is the sweeping rate, the nonlinear parameters c and λ. Equations (6) and (7) reduce to

$$i\frac{\partial}{\partial t}\begin{pmatrix}a\\ b\end{pmatrix}=\boldsymbol{H}\begin{pmatrix}a\\ b\end{pmatrix}, \tag{9}$$

It is clear that the Hamiltonian of the system is modified by the three-body interaction λ.

In order to analyse the behaviour of the energy levels in the nonlinear model when both two- and three-body interactions are take into account, we define $s=|b|^2-|a|^2$. One can obtain $1+2|a|^2|b|^2+2|b|^2=(5-s^2+2s)/2$, $1+2|a|^2|b|^2+2|a|^2=(5-s^2-2s)/2$. Thus, the Hamiltonian becomes

$$\boldsymbol{H}=\begin{pmatrix}\mathcal{C} & v/2\\ v/2 & \mathcal{D}\end{pmatrix}, \tag{10}$$
$$\mathcal{C}=\frac{\gamma}{2}+\frac{c}{2}s+\frac{\lambda}{2}(5-s^2+2s),$$
$$\mathcal{D}=-\frac{\gamma}{2}-\frac{c}{2}d+\frac{\lambda}{2}(5-s^2-2s).$$

The nonlinear eigenstates are then defined as the solution of the time-independent version of Eq. (9), obtained by replacing $i\frac{\partial}{\partial t}$ with the energy ε, we can obtain

$$\Big[\frac{\gamma}{2}+\frac{c}{2}s+\frac{\lambda}{2}(5-s^2+2s)-\epsilon\Big]a=-\frac{v}{2}b, \tag{11}$$
$$\Big[\frac{\gamma}{2}+\frac{c}{2}s+\frac{\lambda}{2}(5-s^2-2s)+\epsilon\Big]b=\frac{v}{2}a. \tag{12}$$

After some elaboration we obtain

$$\lambda s^3+(c+2\epsilon-3\lambda)s+\gamma=0, \tag{13}$$
$$[\gamma+(c+2\lambda)s]^2-[\lambda(s^2-5)+2\epsilon]^2=-v^2. \tag{14}$$

Using Eqs. (13) and (14), we can analyse the influence of the three-body interaction on the energy band structure. It is clear that when $\lambda=0$, Eqs. (13) and (14) reduce to the corresponding result of Ref. [5]. We can expect that three-body interaction should have significant effect on the eigenstates of the system.

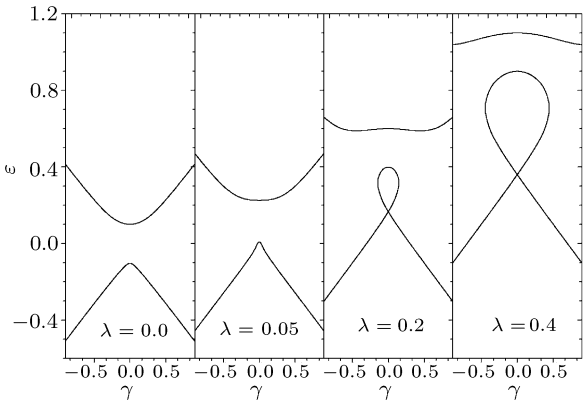


Fig. 1. Energy levels for different λ with $v=0.2$ and $c=0.1$.

Figure 1 shows the energy levels for different λ with $c=0.1$ and $v=0.2$. We can see that the energy band structures are influenced dramatically by the three-body interaction. It is shown that for a fixed $c=0.1<v=0.2$ a critical value $\lambda_0=0.05$ exists. When $\lambda>\lambda_0$, a loop structure appears at the tip $\gamma=0$ of the lower level. The loop width increases with λ. More interestingly, one can find from

Fig. 1 that the effect of three-body interaction on energy levels exhibits some new characters. The energy levels are shifted upward as λ increases. The upper level around $\gamma = 0$ is also charged dramatically. The upper level around $\gamma = 0$ is shifted upward quickly and a flatter upper level occurs with increasing λ. These characters do not occur when $\lambda = 0$.[5] One can expect that the appearance of the loop structure at the lower energy level should be related to both c and λ. The regions in which the loop structure can appear in $c - \lambda$ plane are shown in Fig. 2 for different v. Figure 2 also shows that for fixed v and λ, the loop structure can occur only when $c > c_0$. The critical value c_0 decreases linearly with increasing λ, the region in which the loop structure can be excited is shifted upward in the $c_0 - \lambda$ plane with increasing v. The linearly variation of c_0 against λ can be demonstrated as follows. According to the results reported in Ref. [16], the loop width should be $W = ((c + 2\lambda)^{\frac{2}{3}} - v^{\frac{2}{3}})^{\frac{3}{2}}$. When $W \to 0$, we have $c_0 = -2\lambda + v$, i.e., the critical value c_0 decreases linearly with increasing λ. From Figs. 1 and 2, one can conclude that the three-body interaction has significant effects on the eigenstates.

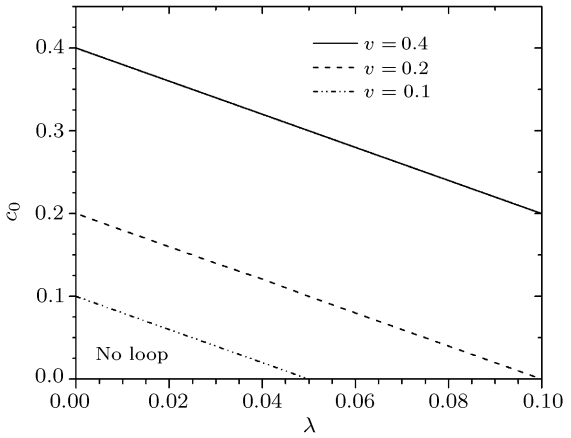


Fig. 2. Critical value c_o as a function of λ at different v.

It is known that the loop structure appears at $\gamma = 0$, breaking down the adiabatic evolution even in the adiabatic limit. This breakdown of adiabatic evolution can be confirmed by numerical calculation of the tunnelling probability directly from Eq. (9). One can also expect that the tunnelling probability should be modified by the three-body interaction. The numerical results of the tunnelling probability for different three-body interaction strengths λ are shown in Fig. 3, with $v = 0.2$ and $c = 0.1$. We can see that the tunnelling probability increases with λ. The tunnelling probability for $\lambda > 0.05$ is not zero in the adiabatic limit $\alpha \to 0$. At the critical point $\lambda_0 = 0.05$, the transition probability seems to vanish with α with a nonzero slope.

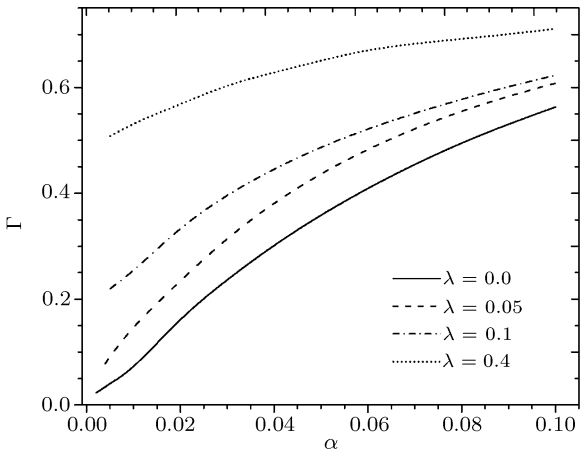


Fig. 3. Numerical results of the tunnelling probability as a function of α for different λ ($v = 0.2$ and $c = 0.1$).

The tunnelling probability can be estimated theoretically. With $a = |a|e^{i\theta_a}$, $b = |b|e^{i\theta_b}$, $s = |b|^2 - |a|^2$ and the relative phase $\theta = \theta_b - \theta_a$, Eqs. (6) and (7) can be reduced to

$$\frac{d}{dt}s(t) = -v\sqrt{1 - s^2(t)}\sin\theta(t), \tag{15}$$

$$\frac{d}{dt}\theta(t) = \gamma + (c + 2\lambda)s(t) + \frac{vs(t)}{\sqrt{1 - s^2(t)}}\cos\theta(t). \tag{16}$$

It is clear that, when $\lambda = 0$, Eqs. (15) and (16) reduce to the result reported in Ref. [7]. On the other hand, because of the three-body interaction, the coefficient of the nonlinear term in Eq. (16) is modified to $c + 2\lambda$. Thus, according to the results of Ref.[7], the tunnelling probability can be easily obtained for two limiting cases.

When $(c+2\lambda)/v < 1$, the tunnelling probability Γ is thus found to be

$$\Gamma \sim \exp\Big(-q\frac{\pi v^2}{2\bar{\alpha}}\Big), \tag{17}$$

where the factor in the exponent is given by

$$q = \frac{4}{\pi}\int_0^{\mathcal{E}}(1 + x^2)^{\frac{1}{4}}\Big[\frac{1}{(1 + x^2)^{\frac{3}{2}}} - \frac{c + 2\lambda}{v}\Big]^{\frac{3}{2}}dx,$$

$$\mathcal{E} = \sqrt{(v/(c + 2\lambda))^{\frac{2}{3}} - 1}. \tag{18}$$

When $(c + 2\lambda)/v \gg 1$, the tunnelling probability Γ is given by

$$\Gamma = 1 - |b|^2_{-\infty} = 1 - \Big(\frac{v}{2}\Big)^2\Big|\int_{-\infty}^{+\infty}dt \cdot \exp\Big(-\frac{i}{2}\bar{\alpha}(t - t_0)^2\Big)\Big|^2 = 1 - \frac{\pi v^2}{2\bar{\alpha}}, \tag{19}$$

where the factor $\bar{\alpha}$ satisfies

$$\bar{\alpha} = \alpha + 2(c + 2\lambda)\Big(\frac{v}{2}\Big)^2\sqrt{\frac{\pi}{\bar{\alpha}}}. \tag{20}$$

It is clear that when $\lambda = 0$, the results given by Eqs. (17)–(20) reduce to the corresponding results of Ref. [5].

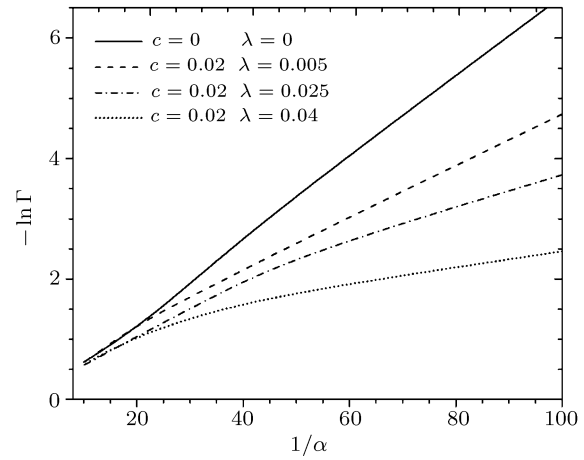


Fig. 4. Numerical results of the exponential dependence of the tunnelling probability as a function of $1/\alpha$ for different λ. The results are for subcritical values of the interaction $(c + 2\lambda) < v = 0.2$.

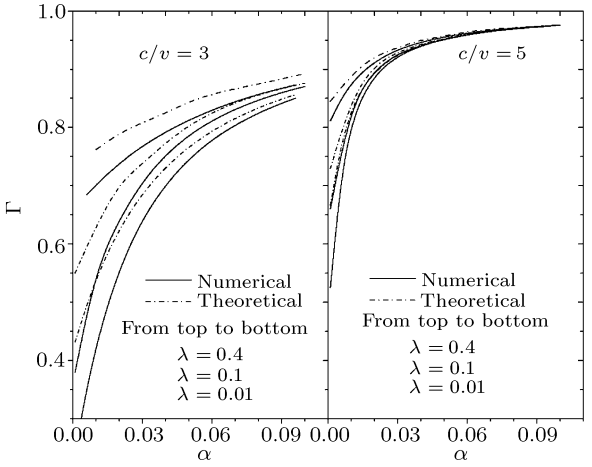


Fig. 5. Tunnelling probability as a function of α for different λ with $(c + 2\lambda) > v$.

For the case of $(c + 2\lambda)/c < 1$, the numerical results of the tunnelling probability for different three-body interaction strength λ are shown in Fig. 4 with $v = 0.2$ and $c = 0.02$. We can see that the tunnelling probability vanishes exponential with α. From the slope of the curves, we can find that for small α, $\Gamma \sim \exp[-q(\pi v^2/2\alpha)]$.

For the case of $(c + 2\lambda)/c \gg 1$, the tunnelling probability given by both analytical results of Eq. (19) and direct numerical results of Eq. (9) are presented in Fig. 5. The results indicate that the tunnelling probability increases with λ when c and v are fixed. It is also shown that the analytical results are in good agreement with the numerical results, especially for larger $(c + 2\lambda)/c$.

In summary, we have investigated the tunnelling properties of BEC with repulsive two- and three-body interactions immersed in an one-dimensional accelerating optical lattice. With the mean-field theory and two-mode approximation, the nonlinear eigenstates and the tunnelling properties are investigated both analytically and numerically. The results shown that the eigenstates and the tunnelling probability are influenced by the three-body interaction dramatically.

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