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General and Optimal Scheme for Local Conversion of Pure States *

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We present general and optimal schemes for local conversion of pure states, via one specific example. First, we give the general solution of the doubly stochastic matrix. Then, we find the general and optimal positive-operator-valued measure (POVM) to realize the local conversion of pure states. Lastly, the physical realization of the POVM is discussed. We show that our scheme has a more general and better effect than other schemes.

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Pure state conversion is a fundamental problem in the study of quantum information. It was first introduced by Bennett *et al.*[1−3] Nielsen[4] gave the necessary and sufficient conditions for entanglement transformation and an algorithm for pure states conversion using only local operation and classical communication. Hardy[5] described a conversion algorithm in terms of cutting and pasting of areas and a coloring problem. Jensen and Schack[6] have proposed another valuable algorithm using the theories of majorization and doubly stochastic matrix. With this algorithm, the local conversion of pure states can be achieved deterministically.

Although Jensen's scheme is useful, the doubly stochastic matrix and the corresponding positive-operator-valued measure (POVM) scheme he gave is not the optimal one. In this Letter, we first present the deduction of the doubly stochastic matrix in detail. Then, general and optimal schemes of POVM are proposed to realize the local conversion of pure states. Lastly, we discuss the physical realization of the POVM.

Consider the same case in Ref. [6]. Alice and Bob each possess a three-dimensional quantum system, with orthonormal bases denoted by $|1\rangle$, $|2\rangle$ and $|3\rangle$. Suppose Alice and Bob have the following initial state $|\phi\rangle$ (or $|\phi\rangle_{AB}$) and final state $|\psi\rangle$ (or $|\psi\rangle_{AB}$).

$$|\phi\rangle = \sqrt{\frac{2}{5}}\,|11\rangle + \sqrt{\frac{1}{4}}\,|22\rangle + \sqrt{\frac{7}{20}}\,|33\rangle, \qquad (1)$$

$$|\psi\rangle = \sqrt{\frac{3}{5}}\,|11\rangle + \sqrt{\frac{3}{10}}\,|22\rangle + \sqrt{\frac{1}{10}}\,|33\rangle. \qquad (2)$$

We will present a scheme to realize the conversion from $|\phi\rangle$ to $|\psi\rangle$. Let α and β be the vectors of the eigenvalues of the reduced density operator of $|\phi\rangle\langle\phi|$ and $|\psi\rangle\langle\psi|$ respectively, $\alpha = (2/5, 1/4, 7/20)^T$ and $\beta = (3/5, 3/10, 1/10)^T$.

It can be inferred from the theory of majorization[7] that β majorizes α, and this follows that a doubly stochastic matrix D exists and satisfies

$$\alpha = D\beta. \qquad (3)$$

An $n\times n$ matrix $D = (D_{ij})$ is called doubly stochastic, if $(D_{ij} \geq 0)$ for all i and j, and

$$\sum_{i=1}^{n} D_{ij} = 1, \quad \text{for all} \quad j, \qquad (4)$$

$$\sum_{j=1}^{n} D_{ij} = 1. \quad \text{for all} \quad i. \qquad (5)$$

Here Eq. (3) can be rewritten in the form

$$\begin{pmatrix} 2/5 \\ 1/4 \\ 7/20 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} \begin{pmatrix} 3/5 \\ 3/10 \\ 1/10 \end{pmatrix}. \qquad (6)$$

According to Eq. (4)–(6), using basic algebra calculation, we can reach the general solution of D,

$$\begin{aligned} D_{11} &= \frac{1}{5}(3 - 2D_{12}), \quad D_{12} = D_{12}, \\ D_{13} &= \frac{1}{5}(2 - 3D_{12}), \quad D_{21} = D_{21}, \\ D_{22} &= \frac{1}{4}(3 - 10D_{21}), \quad D_{23} = \frac{1}{4}(1 + 6D_{21}), \\ D_{31} &= \frac{1}{5}(2 + 2D_{12} - 5D_{21}), \\ D_{32} &= \frac{1}{4}(1 - 4D_{12} + 10D_{21}), \\ D_{33} &= \frac{1}{20}(7 + 12D_{12} - 30D_{21}). \end{aligned} \qquad (7)$$

From Eq. (7), we can see that the doubly stochastic matrix D is not unique. In fact, it is easy to prove that the doubly stochastic matrix is always not unique when $n \geq 3$. Here, with different values of D_{12}

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and D_{21}, different D would be presented. If we set $D_{12} = 2/3$ and $D_{21} = 1/6$, we can reach the matrix

$$\begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix},$$

which is just the special example given in Ref. [6]. Now we give two different examples. If we set $D_{12} = 0, D_{21} = 0$ or $D_{12} = 1/2, D_{21} = 1/5$, we can obtain the matrix

$$\begin{pmatrix} 3/5 & 0 & 2/5 \\ 0 & 3/4 & 1/4 \\ 2/5 & 1/4 & 7/20 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2/5 & 1/2 & 1/10 \\ 1/5 & 1/4 & 11/20 \\ 2/5 & 1/4 & 7/20 \end{pmatrix}.$$

By analogy, we can reach more doubly stochastic matrices, which satisfy $\alpha = D\beta$.

Based on Birkhoff's theorem,[7] a doubly stochastic matrix D can be written as a convex combination of permutation matrices, i.e. $D = \sum p_i P_i$, where $p_i \geq 0$, $\sum p_i = 1$ and P_i is permutation matrix.

Using the same method as in Ref. [6], the POVM can be defined as

$$A_i = p_i \mathrm{diag}\big((P_i\beta)_j/\alpha_j\big), \quad j = 1, 2, 3. \tag{8}$$

Following A_i is applied on the state of $|\phi\rangle$, the vector of the eigenvalues of the reduced density operator of the initial state turns to be $A_i\alpha$, and p_i is the probability of the outcome.

After normalization, the post-measurement states are just $P_i\beta$. To reach the target state β, Alice permutes the bases according to the permutation matrix P_i, and she communicates the permutation matrix P_i with Bob, who then performs the same permutation on his bases. This is the course known as the local operation and classical communication (LOCC). Thus, the local conversion of pure state is realized.

We have just illustrate a general scheme to realize the conversion of pure states locally. The states α and β are specific, however the scheme is general. The scheme is suitable for arbitrary states only if β majorizes α. When we choose states α and β, we do not lose the generality.

As discussed above, it can be noticed that the doubly stochastic matrix D is not unique. With different matrix D, different schemes of POVM would be constructed. Now a question arises: which scheme is the optimal one?

On the one hand, the more the entries of zero the matrix D has, the less the permutation matrices D would be composed of, and the less the POVM operators would be constructed. It can be analysed from Eq. (7) that the maximal number of entries of zero D can have is two.

On the other hand, judging from Eq. (7), we can find out that only six matrices with two entries equal to zero exist. As is shown in the following, any one of these matrices can be broken up into three permutation matrices, and some of the permutation matrices may be unit matrices:

$$\begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{2}{5} & \frac{1}{4} & \frac{7}{20} \end{pmatrix} \begin{pmatrix} \frac{2}{5} & \frac{1}{2} & \frac{1}{10} \\ \frac{1}{5} & \frac{1}{4} & \frac{11}{20} \\ \frac{2}{5} & \frac{1}{4} & \frac{7}{20} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{3}{10} & 0 & \frac{7}{10} \\ \frac{11}{30} & \frac{1}{3} & \frac{3}{10} \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{8}{15} & \frac{1}{6} & \frac{3}{10} \\ \frac{3}{10} & 0 & \frac{7}{10} \\ \frac{1}{6} & \frac{5}{6} & 0 \end{pmatrix}.$$

Among these six matrices, the one that is composed of unit matrix with the maximal coefficient would be optimal one. Since for the POVM operator constructed based on unit matrix, the post-measurement state is just the target state, and no work needs to do in the course of LOCC. Among these six matrices, the fifth is the optimal matrix, which has the maximal coefficient of unit matrix, 1/2. It can be written as

$$\begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 3/4 & 1/4 \\ 1/2 & 0 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{9}$$

The corresponding POVM can be constructed as

$$\begin{aligned} A_1 &= \frac{1}{2}\mathrm{diag}\Big(\frac{3/5}{2/5}, \frac{3/10}{1/4}, \frac{1/10}{7/20}\Big) \\ &= \mathrm{diag}\Big(\frac{3}{4}, \frac{3}{5}, \frac{1}{7}\Big), \\ A_2 &= \frac{1}{4}\mathrm{diag}\Big(\frac{3/10}{2/5}, \frac{1/10}{1/4}, \frac{3/5}{7/20}\Big) \\ &= \mathrm{diag}\Big(\frac{3}{16}, \frac{1}{10}, \frac{3}{7}\Big), \\ A_3 &= \frac{1}{4}\mathrm{diag}\Big(\frac{1/10}{2/5}, \frac{3/10}{1/4}, \frac{3/5}{7/20}\Big) \\ &= \mathrm{diag}\Big(\frac{1}{16}, \frac{3}{10}, \frac{3}{7}\Big). \end{aligned} \tag{10}$$

It can be inferred from Eq. (10) that each A_i is a positive matrix and the sum of A_i is the unit matrix, namely $\sum A_i = I$. Thus, $\{A_i\}$ defines a POVM.

The post-measurement state is

$$\begin{aligned} &A_1\alpha = (3/10, 3/20, 1/20) \to (3/5, 3/10, 1/10), \\ &A_2\alpha = (3/40, 1/40, 3/20) \to (3/10, 1/10, 3/5), \\ &A_3\alpha = (1/40, 3/40, 3/20) \\ &\qquad \to (1/10, 3/10, 3/5), \end{aligned} \tag{11}$$

where $\rightarrow$ indicates the normalization. For A_1, we do not need to do anything. For A_2 and A_3 Alice permutes the bases according to the second and third permutation matrix in Eq. (9), and she communicates the permutation matrices with Bob, who then performs the same permutation on his bases.

In an optimal scheme, we obtain the unit matrix P_i with the maximal probability. In other words, in an optimal scheme, we have the maximal probability to reach the target state and next we do not need to do any thing. This is the real meaning of the "optimal scheme".

Evidently, A_i is not unitary, which can not be realized physically. Next we show how to realize POVM measurements physically by tensor product extension.[8] According to Neumark's theorem, any POVM measurement performed on any quantum system S can always be realized by extending the Hilbert space H_S to a larger space, and performing a conventional orthogonal measurement in the larger space. To do this, we introduce an ancillary system C, and perform an orthogonal measurement on system $|\phi\rangle_{AB}\otimes C$. Specifically, to realize the POVM measurement described in Eq. (11), we introduce a three-state ancillary system C having an orthonormal basis $\{|i\rangle_C, i = 1, 2, 3\}$. Define an operator U acting on $|\phi\rangle_{AB}$ and C by

$$U|\phi\rangle_{AB} \otimes |1\rangle_C = \sum_i A_i|\phi\rangle_{AB} \otimes |i\rangle_C, \qquad (12)$$

where $\sum_i A_i = I$, and U preserves inner products, therefore can be extended to a unitary operator. After performing the unitary operation, an orthogonal measurement is performed on the ancillary system C. If the result is $|i\rangle_C$, $|\phi\rangle_{AB}$ is projected into the corresponding state $P_i\beta$ with probability p_i, thus the POVM is realized physically.

It must be emphasized that our scheme is very constructive. (i) With the example taken in this study, we give a general scheme to construct different POVMs to realize the conversion of pure state. This method can be used extensively. Once the initial and final states satisfy the condition that the final states majorize initial states, a similar scheme can be constructed to convert the initial state to the final state. (ii) In this study, a new criteria is proposed to evaluate which is the optimal scheme. Under an optimal scheme, we have the maximal probability to obtain the target state. (iii) The physical realization of the POVM is discussed, so we can make sure that our scheme can be realized physically.

In conclusion, we have presented general and optimal schemes of POVM to realize the local conversion of pure states. Compared with the algorithm in Ref. [6], our scheme is more general and better, and it may be useful for the future study of pure state conversion.

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