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# Exact Solution of Klein–Gordon Equation by Asymptotic Iteration Method *

Eser Olğar**

*University of Gaziantep, Engineering of Physics Department, Gaziantep, Turkey*

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*Using the asymptotic iteration method (AIM) we obtain the spectrum of the Klein–Gordon equation for some choices of scalar and vector potentials. In particular, it is shown that the AIM exactly reproduces the spectrum of some solvable potentials.*

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In relativistic quantum mechanics, the particle interaction should be described by either the Klein–Gordon (KG) equation or Dirac equation.[1,2] Recently, there have been many studies on the KG equation with various types of potentials by using different methods to describe the relativistic effects.[3−15] Interestingly, in most of the studies, equal scalar $S(x)$ and vector $V(x)$ potentials have been considered in order to obtain the bound-state solutions of the KG equation. On the other hand, there have been very few studies with other choices of $V(x)$ and $S(x)$.[16−19]. Dutra *et al.*[19] defined a transformation between vector and scalar potentials as $V(x) = V_0 + \beta S(x)$, where $V_0$ and $\beta$ are arbitrary constants. This choice is very restricted. Here, to have a bound state solution, $\beta$ must be less then 1, if $V_0 = 0$. In this study, to avoid this restriction, we define a more general transformation between corresponding potentials.

Recently, some methods have been developed in order to solve the Schrödinger equation.[20−22] One of these methods is the asymptotic iteration method (AIM) which has been introduced by Çitçi *et al.*[23] Subsequently, AIM has been used in many physical systems to obtain the whole spectra.[23−32] This method reproduces exact solutions of many exactly solvable and also gives accurate result for the non-solvable potentials such as sextic oscillator, deformed Coulomb potential, etc. Although AIM has been applied to solve the Schrödinger equation for many potentials, there have been very few studies on its application to relativistic equations such as the KG equation.

In this study, we apply the AIM in order to obtain the bound state energy spectrum for the KG equation with a general transformation between vector and scalar potentials. In order to construct exactly solvable potentials for the KG equation, the vector potentials are chosen linear, exponential and inversely linear potentials. It is important that, when the vector potentials are chosen in this form, the KG equation reduces to a Schrödinger-like equation with harmonic oscillator, Morse oscillator and Kratzer potential,[33] respectively.

In this Letter, we deal with the formalism of the one-dimensional KG equation with scalar potential greater than vector potential. A general description of AIM is outlined. Some exactly solvable example are treated by means of AIM. We construct a KG equation including some physical potentials. For this purpose we transform the KG equation in the form of the Schrödinger-like equation, because there have been a large number of papers to tackle the problem in the framework of Schrödinger equation.

Generally, the s-wave Klein–Gordon equation with scalar potential $S(x)$ and vector potential $V(x)$ can be written as[11] ($\hbar = 1, c = 1$)

$$\left\{\frac{d^2}{dx^2} + [E - V(x)]^2 - [m + S(x)]^2\right\} f(x) = 0, \quad (1)$$

where $E$ is the energy, and $m$ is the mass of the particle. Indeed, the radial wave function is expressed as $R(x) = f(x)/x$. When the relation between the potentials is $S(x) \geq V(x)$, it is said that KG equation has a real bound state solutions. When these two potentials is equal to each other, Eq. (1) reduces to the Schrödinger-like equation whose solutions can be obtained with many methods developed in non-relativistic quantum mechanics. In this work, we try to consider the case of $S(x) > V(x)$ for the KG equation. When we rearrange Eq. (1), we obtain

$$\left\{\frac{d^2}{dx^2} - [V_{\text{eff}}(x) - (E^2 - m^2)]\right\} f(x) = 0, \quad (2)$$

where

$$V_{\text{eff}} = [S^2(x) - V^2(x)] + 2[mS(x) + EV(x)].$$

Recently in Ref. [19], the relationship between the corresponding potentials is considered to be

$$V(x) = V_0 + \beta S(x),$$

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**Email: olgar@gantep.edu.tr

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where $V_0$ and $\beta$ are arbitrary constants. In this constraint, in order to have the condition $S(x) > V(x)$, we must take the parameter $\beta$ is less then 1, if $V_0 = 0$. To avoid this restriction, we define the relationship between potentials as

$$S(x) = V(x)(\beta - 1), \quad \beta \geq 0. \tag{3}$$

In this general description of scalar potential, by choosing $\beta$=0, 1, and 2, the scalar potentials lead to the case of $S(x) = -V(x)$, $S(x) = 0$ (purely vector potential), and $S(x) = V(x)$ respectively. The other choices of $\beta$ leads to the required condition for the case of $S(x) > V(x)$. Substituting the expression in Eq. (3) in the s-wave KG equation yields

$$\Big\{\frac{d^2}{dx^2} - [V(x)(V(x)A^2 + B) - \varepsilon]\Big\} f(x) = 0, \tag{4}$$

where $\varepsilon = (E^2 - m^2)$, $A^2 = (\beta^2 - 2\beta)$ and $B = 2[E - m(1 - \beta)]$. After some algebraic simplification, Eq. (4) can be written as

$$\Big\{\frac{d^2}{dx^2} - \Big[\Big[AV(x) + \frac{B}{2A}\Big]^2 - \delta\Big]\Big\} f(x) = 0, \tag{5}$$

where $\delta = \varepsilon + (B/2A)^2$.

Now, after this formalism of the KG equation for unequal vector and scalar potentials, the formulation of the AIM that can be used to obtain the spectra of KG is shown in the following.

The AIM is proposed to solve the second-order differential equations and the details can be found in Ref. [23]. First, we consider the following second order homogeneous differential equation

$$y''(x) = \lambda_0 y'(x) + s_0 y(x), \tag{6}$$

where $\lambda_0$ and $s_0$ are the functions, and $y'(x)$ and $y''(x)$ denotes derivative of $y$ with respect to $x$. It is easy to show that the $(n+2)^{th}$ derivative of the function $y(x)$ can be written as

$$y^{(n+2)}(x) = \lambda_n y'(x) + s_n y(x), \tag{7}$$

where $\lambda_n$ and $s_n$ are given by the recurrence relations

$$\begin{aligned} \lambda_n &= \lambda'_{n-1} + s_{n-1} + \lambda_{n-1}\lambda_0, \\ s_n &= s'_{n-1} + \lambda_{n-1} s_0. \end{aligned} \tag{8}$$

If we have, for sufficiently large $n$,

$$\frac{\lambda_n}{s_n} = \frac{\lambda_{n-1}}{s_{n-1}} = \alpha(x), \tag{9}$$

then the solution to Eq. (6) can be written as[23]

$$y(x) = \exp\Big(-\int^x \alpha dt\Big)\Big[C_1 + C_2 \int^x \exp\Big(\int^s (\lambda_0 + 2\alpha) dt\Big) ds\Big]. \tag{10}$$

In calculating the parameters in Eq. (8), for $n = 0$, we take the initial conditions as $\lambda_{-1} = 1$ and $s_{-1} = 0$[26] and $\Delta_n(x) = 0$ for

$$\Delta_n(x) = \lambda_n(x) s_{n-1}(x) - \lambda_{n-1}(x) s_n(x), \tag{11}$$

where $\Delta_n(x)$ is the termination condition in Eq. (9).

We shall now obtain exact solutions to Eq. (2) for different choices of $S(x)$ and $V(x)$.

We first consider the linear potential form of vector potential. Let $V(x) = x$, then the scalar potential becomes $S(x) = x(\beta - 1)$. After substituting these potentials the corresponding KG equation becomes

$$\Big\{\frac{d^2}{dx^2} - [(Ax + B/2A)^2 - \delta]\Big\} f(x) = 0. \tag{12}$$

After changing of variables $y = Ax + B/2A$, the second-order differential Eq. (12) yields

$$\Big\{\frac{d^2}{dy^2} - [y^2 - \delta]\Big\} f(y) = 0, \tag{13}$$

which is the Klein–Gordon Harmonic oscillator potential. We note that Eq. (12) is in fact a Schrödinger equation with an energy dependent potential.

Now, at this point we apply the AIM. In the limit of large $y$, the asymptotic solutions to Eq. (13) can be taken as any power of $y$ times a decreasing Gaussian. Therefore, it should has a solution in the form of 'normalized' wavefunctions

$$f(y) = \exp(-y^2/2)\chi(y), \tag{14}$$

where the functions $f(y)$ are to be found by means of the iteration procedure. Substituting Eq. (14) into Eq. (13) yields

$$\chi''(y) = 2y\chi'(y) + (1 - \delta)\chi(y), \tag{15}$$

which is now amenable to an AIM solution. By comparing Eq. (15) with Eq. (6), we can write the $\lambda_0(y)$ and $s_0(y)$ values and by means of Eq. (9), we may calculate $\lambda_n(y)$ and $s_n(y)$. These functions for some $n$ values are

$$\begin{aligned} &\lambda_0 = 2y, \quad s_0 = (1 - \delta), \\ &\lambda_1 = 3 - \delta + 4y^2, \quad s_1 = 2(1 - \delta)y, \\ &\lambda_2 = 8y + 2(1 - \delta)y + 2y(3 - \delta + 4y^2), \\ &s_2 = 2(1 - \delta) + (1 - \delta)(3 - \delta + 4y^2), \\ &\qquad \cdots \end{aligned}$$

Combining these results with the quantization condition given by Eq. (13) yields

$$\begin{aligned} &\frac{s_0}{\lambda_0} = \frac{s_1}{\lambda_1} \Longrightarrow \delta_0 = 1, \quad \frac{s_1}{\lambda_1} = \frac{s_2}{\lambda_2} \Longrightarrow \delta_1 = 3, \\ &\frac{s_2}{\lambda_2} = \frac{s_3}{\lambda_3} \Longrightarrow \delta_2 = 5, \qquad \cdots \end{aligned}$$

If we generalize these expressions, we find

$$\delta_n = 2n+1, \quad n = 0, 1, 2, \cdots$$

If we go back to the definitions of parameters $\delta$, $\varepsilon$, $A$ and $B$, it is easy to obtain

$$E^2 - m^2 + \frac{[2(E - m(1-\beta))]^2}{4(2\beta - \beta^2)} = 2n+1, \qquad (16)$$

which is exactly the same as the eigenvalue equation obtained in Ref. [18]. From Eq. (16), one can find that

$$E_n^{\pm} = \frac{(-1+\beta)m \pm \sqrt{\gamma_n}}{-1+(-2+\beta)\beta}, \qquad (17)$$

where $\gamma_n = (-2+\beta)\beta(-1-2n+(-2+\beta)\beta(1+2m^2+2n))$.

In this example, we deal with the exponential form of scalar potential. Let $V(x) = -e^{-\alpha x}$ and then $S(x) = -e^{-\alpha x}(1-\beta)$. After substituting these expressions into Eq. (5), we obtain the differential equation

$$\Big\{\frac{d^2}{dx^2} - [(A^2 e^{-2\alpha x} - Be^{-\alpha x}) - \xi]\Big\} f(x) = 0, \qquad (18)$$

where $\xi = -\varepsilon$, and Eq. (18) is the equation form of the Klein–Gordon Morse potential. We note that as in the previous case, this can be interpreted as a Schrödinger equation with an energy dependent potential. After changing of variables $y = e^{-\alpha x}$, we obtain

$$\Big\{\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} + \Big[\frac{B}{\alpha^2 y} - \frac{A^2}{\alpha^2} + \frac{\xi}{\alpha^2 y^2}\Big]\Big\} f(y) = 0. \qquad (19)$$

Now, we reach a position that the differential equation is suitable for applying AIM. Therefore, Eq. (19) should have a solution in the form of the normalized wavefunctions,

$$f(y) = y^{\sqrt{\xi}/\alpha} \exp\Big(-\frac{A}{\alpha}y\Big)\chi(y). \qquad (20)$$

Substituting Eq. (20) into Eq. (19), one obtains

$$\chi''(y) = \Big[\frac{-2\sqrt{\xi} - \alpha + 2Ay}{\alpha y}\Big]\chi'(y) + \Big[\frac{-B + 2A\sqrt{\xi} + A\alpha}{\alpha^2 y^2}\Big]\chi(y), \qquad (21)$$

which is now amenable to an AIM solution. By comparing Eq. (21) with Eq. (6), we can write the $\lambda_0(y)$ and $s_0(y)$ values and by means of Eq. (9), we may calculate $\lambda_n(y)$ and $s_n(y)$. These functions for some low-lying values of $n$ are

$$\lambda_0 = -\frac{2\sqrt{\xi} + \alpha - 2Ay}{\alpha y}, \quad s_0 = \frac{-B + A(2\sqrt{\xi} + \alpha)}{\alpha^2 y},$$
$$\lambda_1 = \frac{1}{\alpha^2 y^2}[4\xi - By + 4A^2y^2 - 3Ay\alpha + 2\alpha^2 + \sqrt{\xi}(-6Ay + 6\alpha)],$$
$$s_1 = -\frac{1}{\alpha^3 y^2}(\sqrt{\xi} - Ay + \alpha)[B - A(2\sqrt{\xi} + \alpha)],$$
$$\cdots$$

Similar to the previous example, by combining these results with the quantization condition yields

$$\frac{s_0}{\lambda_0} = \frac{s_1}{\lambda_1} \Longrightarrow \xi_0 = \Big(\frac{1}{2}\alpha - \frac{B}{2A}\Big)^2,$$
$$\frac{s_1}{\lambda_1} = \frac{s_2}{\lambda_2} \Longrightarrow \xi_1 = \Big(\frac{3}{2}\alpha - \frac{B}{2A}\Big)^2,$$
$$\frac{s_2}{\lambda_2} = \frac{s_3}{\lambda_3} \Longrightarrow \xi_2 = \Big(\frac{5}{2}\alpha - \frac{B}{2A}\Big)^2,$$
$$\cdots$$

From the above expressions, it is possible to write the general formula of $\xi$ as

$$\xi_n = \Big(\Big(n + \frac{1}{2}\Big)\alpha - \frac{B}{2A}\Big)^2, \quad n < \Big(\frac{B}{\alpha^2} - \frac{1}{2}\Big), \qquad (22)$$

where the relation of $n$ is obtained by comparing our parameters with that of in Refs. [35,36].If we go back to the definitions of parameters $\xi$, and $\varepsilon$, it can be shown that

$$E_n^2 - m^2 = -\Big(\Big(n + \frac{1}{2}\Big)\alpha - \frac{B}{2A}\Big)^2, \qquad (23)$$

which is exactly the same as the eigenvalue equation obtained in Ref. [18] through a proper choice of parameters.

The last example is the Kratzer potential that has an important role in quantum mechanical systems. In order to obtain the required potential form in the KG equation, let us take $V(x) = 1/x$ and then $S(x) = 1/x(1-\beta)$. Substituting them into Eq. (5), we reach the differential equation

$$\Big\{\frac{d^2}{dx^2} - \Big[\Big(\frac{A^2}{x^2} - \frac{B}{x}\Big) - \xi\Big]\Big\} f(x) = 0, \qquad (24)$$

where $\xi = -\varepsilon$, and Eq. (24) is the equation form of the Klein–Gordon Kratzer potential. In this position, to obtain the form of differential equation (6), we propose the wavefunction

$$f(x) = x^A \exp(-\sqrt{\xi}x)\chi(x). \qquad (25)$$

Substituting Eq. (20) into Eq. (19) yields

$$\chi''(x) = 2\Big[\frac{\sqrt{\xi}x - A}{x}\Big]\chi'(x) + \Big[\frac{A + (B + 2A\sqrt{\xi})x}{x^2}\Big]\chi(x). \qquad (26)$$

Comparing Eq. (26) with Eq. (6), we can write the $\lambda_0(x)$ and $s_0(x)$ values and we may calculate $\lambda_n(x)$ and $s_n(x)$ from Eq. (9). The first two terms are

$$\lambda_0 = (2\sqrt{\xi}y - 2A)/x,$$
$$s_0 = (A + Bx + 2A\sqrt{\xi}x)/x^2,$$

$$\lambda_1 = \frac{1}{y^2}[4A^2 + A(3 - 6\sqrt{\xi}x) + x(B + 4\xi x)],$$
$$s_1 = \frac{1}{x^3}[Bx(-1 + 2\sqrt{\xi}x) - 2A^2(1 + 2\sqrt{\xi}x) - 2A(1 + Bx - 2\xi x^2)],$$
$$\cdots$$

Similarly, from the quantization condition, we obtain

$$\frac{s_0}{\lambda_0} = \frac{s_1}{\lambda_1} \Longrightarrow \xi_0 = \Big(\frac{B}{2A}\Big)^2,$$
$$\frac{s_1}{\lambda_1} = \frac{s_2}{\lambda_2} \Longrightarrow \xi_1 = \Big(\frac{B}{2(1+A)}\Big)^2,$$
$$\frac{s_2}{\lambda_2} = \frac{s_3}{\lambda_3} \Longrightarrow \xi_2 = \Big(\frac{B}{2(2+A)}\Big)^2,$$
$$\cdots$$

The general formula of $\xi$ for $n$ values can be written as

$$\xi_n = \Big(\frac{B}{2(n+A)}\Big)^2, \quad n = 0, 1, 2, \cdots \tag{27}$$

The eigenvalues in Eq. (27) is exactly the same as the values for Schrödinger equation of Kratzer potential with a suitable choice of parameters.[34] From the definitions of parameters $\xi$ and $\varepsilon$, it is easy to obtain the energy expression

$$E_n^2 - m^2 = -\Big(\frac{B}{2(n+A)}\Big)^2, \tag{28}$$

for the Klein–Gordon-Kratzer potential.

In summary, we have obtained the whole spectrum of some exactly solvable potential in the KG equation by using the AIM method. We have found that the general transformation between vector and scalar potentials is more useful to show the relation between each other.

By choosing appropriate vector potentials, most of the exactly solvable and quasi-exactly solvable potentials can be constructed in this way. We feel that it could be interesting to apply the AIM to other relativistic equations, e.g., the Dirac equation.

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