Dependence of the Average Lorentz Factor on Temperature in Relativistic Plasmas *

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For the relativistic plasma, how to fix the Lorentz factors of the particles is an important but difficult problem. We resolve this problem by demonstrating the exact relation between the average Lorentz factor and temperature in relativistic plasmas. A rather simple relation is also obtained for the ultra-relativistic case.

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The interactions between the intense laser and matter can produce energetic particles, such as MeV electrons,[1] which may produce relativistic plasmas. Much theoretical and experimental attention has increasingly been devoted to the interaction mechanism.[2−5] The physics of relativistic laser-plasmas has renewed the interest in laser-plasma interaction in the past 10 years.[3] Mora *et al.* have theoretically studied the physics of relativistic laser-plasma.[6−8] Pegoraro and Porcelli[9] have derived dispersion relations of electrostatic plasma waves from the relativistic fluid equations and relativistic Vlasov equation; and Jan Bergman and Bengt Eliasson have obtained linear wave dispersion laws in unmagnetized relativistic plasma.[10] Depending on the physical situation, the relativistic plasma can be characterized by a set of parameters, such as the Lorentz factor γ, the plasma frequency ω_p, and the Debye length λ_D, etc. It is difficult to measure some parameters including Lorentz factor γ (or average value $\langle\gamma\rangle$) in relativistic plasmas. Consequently, it is important for us to find out the relationship between γ and a measurable temperature T_e. However, most electronic Lorentz factors γ are different from others in relativistic plasma due to the velocity distribution. Using relativistic kinetic theory to study relativistic plasma, one must also take the relation between the average Lorentz factor $\langle\gamma\rangle$ and temperature T_e into account.

In this Letter, we concentrate mainly on the relationship between the average Lorentz factor and temperature in unmagnetized and homogeneous plasmas with Maxwellian equilibrium distribution functions. Here the electron gas is assumed to be a system of quasi-independent particles in relativistic plasma. We focus on the high temperature range, i.e. $T_e > 5\,\mathrm{keV}$, where the relativistic effects should be taken into account. We derive the exact relation formula between the average Lorentz factor $\langle\gamma\rangle$ and temperature T_e in relativistic plasmas. Based on the ultra-relativistic approximation, we obtain the concise relation formula between the average Lorentz factor $\langle\gamma\rangle$ and temperature T_e, and find the average electronic energy is times of the temperature of the system in the space.

In the relativistic plasma, the energy ε and the momentum vector $\boldsymbol{p}$ of an electron with rest mass m_e are given by

$$\varepsilon = \gamma m_e c^2 = \sqrt{p^2c^2 + m_e^2c^4}, \tag{1}$$

$$\boldsymbol{p} = \gamma\, m_e \boldsymbol{v}, \tag{2a}$$

or

$$p^2 = m_e^2c^2(\gamma^2 - 1), \tag{2b}$$

where $\boldsymbol{v}$ is the electronic velocity vector in the relativistic plasma and c is the speed of light in vacuum. The Lorentz factor γ is

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{3}$$

We assume that the equilibrium distribution is a relativistic Maxwellian distribution in relativistic plasma, i.e. the Juter–Synge distribution[11]

$$f(\gamma) = \frac{\mu e^{-\mu\gamma}}{4\pi m_e^3 c^3 K_2(\mu)}, \tag{4}$$

where μ is a dimensionless parameter, and

$$\mu = \frac{m_e c^2}{k_B T_e}, \tag{5}$$

$K_2(\mu)$ is a modified Bessel function and is given by the second kind modified Bessel function $K_n(x)$, which satisfies the modified Bessel equation

$$x^2y'' + xy' - (x^2 + n^2)y = 0. \tag{6}$$

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$K_n(x)$ can be expressed by[12]

$$K_n(x) = \frac{\sqrt{\pi}\left(\frac{1}{2}x\right)^n}{\Gamma\left(n+\frac{1}{2}\right)\int_1^\infty e^{-xt}(t^2-1)^{n-\frac{1}{2}}dt}. \quad (7)$$

For the isotropic relativistic plasma, the average Lorentz factor $\langle\gamma\rangle$ can be written as

$$\langle\gamma\rangle = \iiint \gamma f(\gamma)d^3p = \int_0^\infty \gamma f(\gamma)4\pi p^2 dp. \quad (8)$$

Taking Eqs. (2b) and (4) into Eq. (8), we can obtain

$$\langle\gamma\rangle = \frac{\mu}{K_2(\mu)}\int_1^\infty \gamma^2\sqrt{\gamma^2-1}e^{-\mu\gamma}d\gamma. \quad (9)$$

Using Eq. (5) and performing this integral, the exact relation formula between the average Lorentz factor $\langle\gamma\rangle$ and temperature T_e in the relativistic plasma is

$$\langle\gamma\rangle = \frac{1}{\frac{m_e^2c^4}{k_B^2T_e^2}K_2\left(\frac{m_ec^2}{k_BT_e}\right)}\left[2I_1\left(\frac{m_ec^2}{k_BT_e}\right) + \frac{m_e^2c^4}{k_B^2T_e^2}K_1\left(\frac{m_ec^2}{k_BT_e}\right) + 3\frac{m_ec^2}{k_BT_e}K_2\left(\frac{m_ec^2}{k_BT_e}\right)\right], \quad (10)$$

where $I_1\left(\frac{m_ec^2}{k_BT_e}\right)$ is a modified Bessel function and is given by the first kind of modified Bessel function $I_n(x)$, which also satisfies the differential equation (6), i.e. another solution of this differential equation.

In the high temperature end of scale, $T_e > 500\,\mathrm{keV}$, the ultra-relativistic approximation can be used. In this case we can assume $p^2 \gg m_e^2c^2$, approximately

$$\varepsilon \approx pc, \quad (11)$$

which neglects the energy for the electronic rest mass, and

$$\gamma^2 = 1 + \frac{p^2}{m_e^2c^2} \approx \frac{p^2}{m_e^2c^2}. \quad (12)$$

In the ultra-relativistic approximation of Eq. (11), the ultra-relativistic plasma can be described by the equilibrium ultra-relativistic Maxwellian distribution[10]

$$f(p) = \frac{c^3e^{-pc/k_BT_e}}{8\pi k_B^3T_e^3}. \quad (13)$$

The average Lorentz factor $\langle\gamma_{ul}\rangle$ in the ultra-relativistic plasma can be written as

$$\langle\gamma_{ul}\rangle = \iiint \gamma f(p)d^3p \approx \iiint \frac{p}{m_ec}f(p)d^3p. \quad (14)$$

For the isotropic ultra-relativistic plasma, from Eq. (14), we can obtain

$$\langle\gamma_{ul}\rangle = \frac{3k_BT_e}{m_ec^2}. \quad (15)$$

Equation (15) shows that the average Lorentz factor $\langle\gamma_{ul}\rangle$ is proportional to the electronic temperature T_e in the ultra-relativistic plasma, and almost coincides with the result of Eq. (10), when $T_e > 500\,\mathrm{keV}$ (see Fig. 2.). Figures 1 and 2 (solid line) show the exact relation between the average Lorentz factor $\langle\gamma\rangle$ and temperature T_e given by Eq. (10) in fully relativistic plasmas. The nonlinear curve for the low end of the temperature is presented in Fig. 1. From Figs. 1 and 2, we can see that curves rises linearity with the increasing temperature. The dot-dot-dashed line in Fig. 2 shows the relation between the average Lorentz factor $\langle\gamma\rangle$ and temperature T_e given by Eq. (15) in ultra-relativistic plasma.

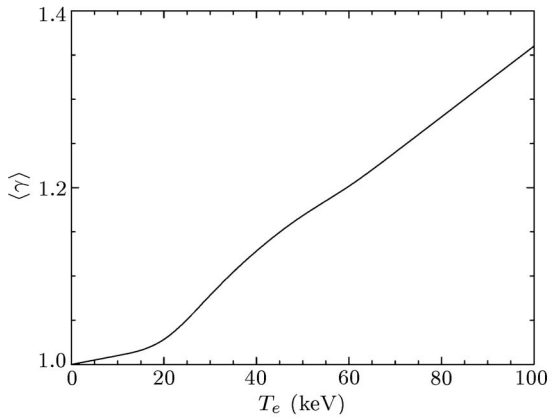

Fig. 1. The relation between the average Lorentz factor and temperature in the lower temperature case.

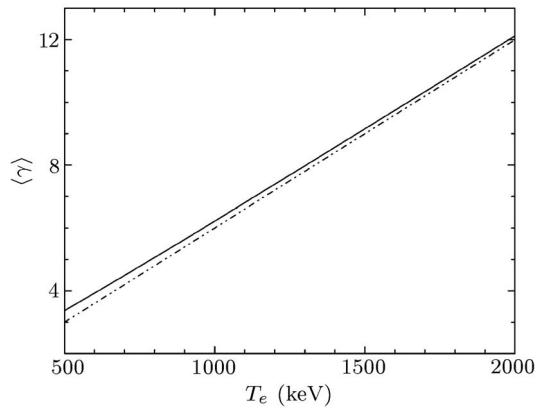

Fig. 2. The relation between the average Lorentz factor and temperature in the ultra-high temperature case.

Moreover, from Eq. (15), we find that the average electronic energy is triple of temperature in the ultra-relativistic plasma in three-dimensional space, i.e.

$$\langle\varepsilon\rangle_3 = 3k_BT_e. \quad (16)$$

For the one-dimensional case, the equilibrium ultra-relativistic Maxwellian distribution expression in ultra-relativistic plasma can be written as

$$f(p) = \frac{c}{2k_B T_e} e^{-pc/k_B T_e}. \tag{17}$$

In this case, the average Lorentz factor $\langle \gamma_{ul} \rangle_{one}$ in ultra-relativistic plasma can be written as

$$\langle \gamma_{ul} \rangle_{one} = \int \gamma f(p) dp. \tag{18}$$

Taking Eq. (17) into Eq. (18) and using Eq. (12), we can obtain the average Lorentz factor $\langle \gamma_{ul} \rangle_{one}$ in the one-dimensional case,

$$\langle \gamma_{ul} \rangle_{one} = \frac{k_B T_e}{m_e c^2}, \tag{19}$$

or

$$\langle \varepsilon \rangle_1 = k_B T_e, \tag{20}$$

where $\langle \varepsilon \rangle_1$ is the average electronic energy in the ultra-relativistic plasma in the one-dimensional case. From Eqs. (16) and (20), we can deduce that the average relativistic kinetic energy is three or one times the temperature of the system in three- or one-dimensional space, which can be regarded as a development from the non-relativistic to the ultra-relativistic limit.

In conclusion, as far as we know, there is not an exact relation formula between the average kinetic energy of relativistic electrons and the temperature of the system, though the particle average kinetic energy $\langle E_k \rangle$ is equal to $\frac{3}{2} k_B T$ in three-dimensional space for non-relativistic cases. As is well known, the relativistic kinetic energy of an electron is E_k (MeV) $= 0.511(\gamma - 1)$. Some researchers, e.g. Refs. [13,14], simply took that the average kinetic energy $\langle E_k \rangle$ for electrons of the system is equal to $k_B T_e$ in the relativistic case, but there was a lack of exact demonstration by relativistic theory. In the present study, based on the relativistic theory and Maxwellian distribution in fully relativistic plasmas, we have investigated the relation between the average Lorentz factor $\langle \gamma \rangle$ and temperature T_e, which is nonlinear in the low temperature range and linear in the high temperature range. Correspondingly, other parameters including $\langle \gamma \rangle$ can be associated with measurable temperature T_e. Additionally, in the ultra-relativistic plasma, we find that the average electronic energy is triple the electron temperature in three-dimensional space, and is equal to the electron temperature in the one-dimensional case.

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