CHIN.PHYS.LETT. Vol. 25, No. 12 (2008) 4189

Periodic Homoclinic Wave of (1+1)-Dimensional Long–Short Wave Equation *

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(Received 22 April 2008)

The exact periodic homoclinic wave of (1+1)D long–short wave equation is obtained using an extended homoclinic test technique. This result shows complexity and variety of dynamical behaviour for a (1+1)-dimensional long–short wave equation.

PACS: 02.30.Jr, 05.45.Yv, 47.11.+j

In recent years, homoclinic orbit and chaos in partial differential equations have attracted extensive attention of mathematicians and physicians. As is well known, the existence of a homoclinic orbit is indicative of chaos for deterministic dynamical systems, see Refs. [1–3]. If homoclinic orbits exist for unperturbed systems, the existence of these orbits, and further chaos, can be studied analytically and numerically under perturbation. Such successes have been achieved for soliton equations which have complete integrability such as Sine–Gordon equations[4−7] and nonlinear Shrödinger equations.[8−11]

There has been a great amount of activity aiming to find methods for exact solutions of nonlinear partial differential equations. In the past decades, there has been significant progress in the development of these methods, such as the Hirota method,[12] Bäcklund transformation,[13] Painlevé expansion,[14] sine–cosine method,[15] homogeneous balance method,[16] homotopy perturbation method,[17] variational method,[18] Adomian method,[19] tanh-function method,[20] algebraic method,[21] Jacobi elliptic function expansion method,[22] F-expansion method,[23] Exp-function method,[24] and so on. However, to date all of the exact solutions including various types of solitary-wave solutions and periodic solutions are functions of the single variable $\xi = kx - \omega t$.

In this Letter, we introduce a new technique, the extended homoclinic test technique, seeking for periodic solitary wave solutions of the integrable equation. The homoclinic test technique is a method used to seek for the homoclinic solution of an integrable system.[25] By using this method, some new periodic solitary-wave solutions for the Boussinesq equation and (1+2)-dimensional integrable system, for example, the Kadomtsev–Petviashvili equation, were obtained and spatiotemporal bifurcation was investigated in recent years.[26−32] The extended homoclinic test technique is an extension of the homoclinic test method: the main difference between the extended homoclinic test technique and the homoclinic test method is using the different test functions to construct the exact solutions. Applying the extended homoclinic test technique to the (1+1)D long–short wave equation, we obtain a periodic homoclinic wave solution which is a new type of wave solution. This result shows that the integrable system possesses richly dynamical behaviour, even if the system is (1+1)-dimensional.

We consider the following long–short wave equations:

$$iS_t + \lambda S_{xx} = LS, \quad L_t = -\alpha(|S|^2)_x, \tag{1}$$

where S is the envelope of the short wave, L is then amplitude of long wave, α and λ are positive constants. Let $S = \lambda^{1/4}u/\sqrt{\alpha}$, $L = v, x = \sqrt{\lambda}x'$, $t = t'$, and substituting x' and t' with x and t respectively. The system (1) can be rewritten as

$$iu_t + u_{xx} = uv, \quad v_t + (|u|^2)_x = 0. \tag{2}$$

Using the transformation of dependent variable

$$u = G/F, \quad v = A - 2(\ln F)_{xx}, \tag{3}$$

where A is a constant, G is a complex valued function and F is a real valued function, Eq. (2) is transformed into the bilinear form

$$(iD_t + D_x^2)G \cdot F = A(G \cdot F), \tag{4}$$

$$D_t D_x(F \cdot F) + c(F \cdot F) = G \cdot G^*, \tag{5}$$

where c is an integration constant.

We suppose that

$$\begin{aligned} G &= e^{-iat}[e^{-p_1(x-\alpha t)} + b_1 \cos p(x+\alpha t) + b_2 e^{p_1(x-\alpha t)}], \\ F &= e^{-p_1(x-\alpha t)} + b_3 \cos p(x+\alpha t) + b_4 e^{p_1(x-\alpha t)}, \end{aligned} \tag{6}$$

where a, p, p_1, α, b_3, and b_4 are real, b_1 and b_2 are complex.

*Supported by the National Natural Science Foundation of China under Grant Nos 10661002 and 10701076, Guangxi Natural Science Foundation under Grant No 0832065, and Yunnan Natural Science Foundation under Grant No 2006A0082M.

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Substituting Eq. (6) into Eq. (4) we can reach an algebraic equations of $e^{p_1(x-\alpha t)}$. Equating the coefficients of all powers of $e^{jp_1(x-\alpha t)}, j = -2,-1,0,1,2$ to zero yields the relations between $A, p, p_1, \alpha, b_1, b_2, b_3, b_4$ as follows:

$$A = a,\ i\alpha p_1(b_3 - b_1) + (b_1 + b_3)(p_1^2 - p^2) = 0, \quad (7)$$

$$i\alpha p(b_3 - b_1) - 2pp_1(b_1 + b_3) = 0, \quad (8)$$

$$i\alpha p_1(b_4 - b_2) + 2p_1^2(b_2 + b_4) - p^2 b_1 b_3 = 0, \quad (9)$$

$$i\alpha p_1(b_1 b_4 - b_2 b_3) + (b_1 b_4 + b_2 b_3)(p_1^2 - p^2) = 0, \quad (10)$$

$$i\alpha p(b_2 b_3 - b_1 b_4) + 2pp_1(b_1 b_4 + b_2 b_3) = 0. \quad (11)$$

From Eq. (7), we have

$$b_1 = \frac{i\alpha p_1 + (p_1^2 - p^2)}{i\alpha p_1 - (p_1^2 - p^2)} b_3. \quad (12)$$

From Eq. (8), we have

$$b_1 = \frac{i\alpha - 2p_1}{i\alpha + 2p_1} b_3. \quad (13)$$

It follows that, from Eqs. (12) and (13),

$$p^2 = 3p_1^2. \quad (14)$$

From Eq. (10), we have

$$b_2 = \left(\frac{i\alpha - 2p_1}{i\alpha + 2p_1}\right)^2 b_4. \quad (15)$$

From Eq. (9), we have

$$b_3^2 = \frac{4(3\alpha^2 - 4p_1^2)}{3(\alpha^2 + 4p_1^2)} b_4. \quad (16)$$

Substituting Eq. (15) into Eq. (11), we know that Eq. (11) is satisfied. In addition, substituting Eq. (6) into Eq. (5) yields

$$c = 1, \quad (2 - 2\alpha(p_1^2 + p^2))b_3 = b_1^* + b_1, \quad (17)$$

$$-8\alpha p_1^2 b_4 - 2\alpha p^2 b_3^2 + 2b_4 = b_2 + b_2^*, \quad (18)$$

$$b_3^2 = b_1 b_1^*, \quad (19)$$

$$\big(2 - 2\alpha(p_1^2 + p^2)\big)b_3 b_4 = b_1 b_2^* + b_1^* b_2, \quad (20)$$

$$b_4^2 = b_2 b_2^*. \quad (21)$$

From Eq. (17), we obtain

$$p_1^2 = \frac{2 - \alpha^3}{4\alpha}. \quad (22)$$

By verifying, we can know Eqs. (13)–(16) and (22) satisfy Eq. (18)–(21).

Combining Eqs. (12)–(16) and Eqs. (21)–(22), we obtain the relations among these parameters

$$A = a,\ c = 1,\ \frac{p^2}{3} = p_1^2 = \frac{2 - \alpha^3}{4\alpha},\ b_1 = \frac{i\alpha - 2p_1}{i\alpha + 2p_1} b_3,$$

$$b_2 = \left(\frac{i\alpha - 2p_1}{i\alpha + 2p_1}\right)^2 b_4, \quad b_3^2 = \frac{4(3\alpha^2 - 4p_1^2)}{3(\alpha^2 + 4p_1^2)} b_4, \quad (23)$$

where $0 < \alpha < \sqrt[3]{2}$.

Substituting Eq. (6) into Eq. (3), we obtain the solution of Eq. (2)

$$\begin{aligned} u = {} & e^{i(\theta - at)}\Big[2\cosh(p_1(x - \alpha t) + \ln\sqrt{b_4} + i\theta) \\ & + \frac{b_3}{\sqrt{b_4}}\cos p(x + \alpha t)\Big]\Big[2\cosh(p_1(x - \alpha t) \\ & + \ln\sqrt{b_4}) + \frac{b_3}{\sqrt{b_4}}\cos p(x + \alpha t)\Big]^{-1}, \\ v = {} & A + 2\Big[2b_3\sqrt{b_4}\cos\Big(\frac{\pi}{3} - p(x + \alpha t)\Big) \\ & \cdot \cosh(p_1(x - \alpha t) + \ln\sqrt{b_4}) + (4b_4 - 3b_3^2)p_1^2\Big] \\ & \cdot \Big\{b_4[2\cosh(p_1(x - \alpha t) + \ln\sqrt{b_4}) \\ & + \frac{b_3}{\sqrt{b_4}}\cos p(x + \alpha t)]^2\Big\}^{-1}, \end{aligned} \quad (24)$$

where $A, p, p_1, \alpha, b_1, b_2, b_3, b_4$, satisfy (23) (Fig. 1). $\dfrac{i\alpha - 2p_1}{i\alpha + 2p_1} = e^{i\theta}$, and $\theta = \tan^{-1}\dfrac{4\alpha p_1}{\alpha^2 - 4p_1^2}$.

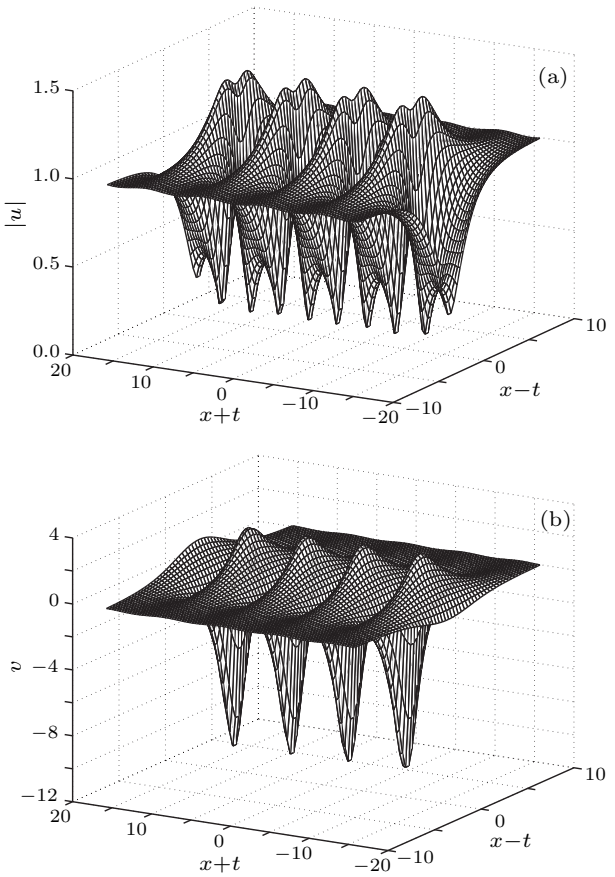

Fig. 1. (a) Behaviour of $|u(x,t)|$ as $b_4 = 1, \alpha = 1$. (b) Behaviour of $v(x,t)$ as $b_4 = 1, \alpha = 1$.

The solution given by Eq. (24) represents a homoclinic wave. In fact, we can prove

$$(u, v) \to (e^{-iat}, A), \text{ as } t \to +\infty,$$
$$(u, v) \to (e^{i(2\theta - at)}, A), \text{ as } t \to -\infty,$$

where $\frac{b_2}{b_4} = e^{2i\theta}$, (e^{-iat}, A) and $(e^{i(2\theta-at)}, A)$ belong to one and the same fixed cycle (e^{-iat}, A) of Eq. (2). The solution expressed by Eq. (24) is also periodic wave with spatiotemporal variable (x, t) because of $u(x, t) = u\Big(x + \frac{\pi}{p}, t + \frac{\pi}{p\alpha}\Big)$, $v(x, t) = v\Big(x + \frac{\pi}{p}, t + \frac{\pi}{p\alpha}\Big)$ for every $(x, t) \in R \times R$. Therefore the solution (u, v) of the long–short wave equation is a two-wave solution containing a periodic wave and a homoclinic wave which have the same speed α, it shows elastic interaction between a left-propagation (backward-direction) periodic wave and right-propagation (forward-direction) homoclinic wave. This is an interesting physical phenomenon for the long–short wave equation; this result shows the interaction of the long wave and short wave described by the long–short wave equation and shows some complexity and variety of dynamical behaviours for the (1+1)-dimensional long–short wave equation.

In addition, from Eq. (24), we find that the obtained solutions $u(x, t)$ and $v(x, t)$ are consistent with the physical meaning of function $u(x, t)$ and $v(x, t)$ of the model equation in Ref. [33].

The authors would like to thank the referees for valuable suggestion and help.

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