Talbot Effect in Three Waveguide Arrays *

LI Zhi(李植)**, ZHOU Hai-Feng(周海峰), YANG Jian-Yi(杨建义), JIANG Xiao-Qing(江晓清)***
Department of Information Science and Electronics Engineering, Zhejiang University, Hangzhou 310027

(Received 4 June 2008)

By taking the coupling between the non-neighbourhood waveguides into account, the coupling characteristic of three waveguide arrays is analysed. The strong coupling equation of three waveguides is dealt with Laplace transform and LU decomposition. The general field evolution equation is obtained by inversion of the Laplace transform. The results show that the self-imaging conditions (Talbot effect) do not satisfy in general. The theoretical predictions are in good agreement with the BPM simulations.

PACS: 42.82.Et, 42.79.Gn, 42.82.−m

Talbot effect, i.e. the self-imaging of a diffraction grating, was first observed by Talbot[1] in 1836. Talbot effect was first introduced when lens-like imaging produced solely by free-space propagation of a diffracted field was observed. Similar phenomena have been observed in many optical devices and systems[1−7] such as gratings,[1,2] multi-mode waveguide,[5] and infinite waveguide arrays.[6] There are many analyses and explanation for Talbot effect in these optical devices. Theoretical analysis indicates that Talbot recurrences in an infinite waveguide array occur only when the period N of the initial pattern (U) belongs to the set $N \in \{1,2,3,4,6\}$.[6] Where N represents the spatial period of the input, i.e. $U_{n+N} = U_n$. Theoretical analysis for Talbot effect in finite waveguide arrays is given when only coupling between neighbourhood waveguides was considered.[7] However, since only neighbourhood coupling was considered, the proposed conditions for Talbot revival do not work in many circumstances.

Taking into account coupling between non-neighbourhood waveguides, in this Letter we discuss the coupling characteristic of three waveguide arrays. We combine the Laplace transform and LU decomposition[8] to calculate the field evolution equation. This method could arrive at an analytic solution.

By using coupled mode theory[9] and coupling between non-neighbourhood waveguides (Fig. 1), the normalized evolution of the amplitude of the optical field in three uniform waveguide arrays satisfies

$$\begin{aligned} iE_1' + \kappa_{1,2}E_2 + \kappa_{1,3}E_3 &= 0, \\ iE_2' + \kappa_{2,1}E_1 + \kappa_{2,3}E_3 &= 0, \\ iE_3' + \kappa_{3,1}E_1 + \kappa_{3,2}E_2 &= 0, \end{aligned} \tag{1}$$

where $E_j' = dE_j/dz$, $j = 1,2,3$, z is the propagation distance, $\kappa_{i,j}$ stands for the coupling coefficient of the field between waveguides i and j. With respect to the periodic structure and boundary conditions,

$$\kappa_{j,j+1} = \kappa_{j+1,j}^*, \quad \kappa_{j,j+2} = \kappa_{j+2,j}^*. \tag{2}$$

For non-loss dielectric waveguide, the coupling coefficients are positive as expected, i.e. $\kappa_{i,j} \geq 0$. Equation (1) has a Hamiltonian structure

$$iE_j' + \frac{\partial H}{\partial E_j^*} = 0, \tag{3}$$

where

$$\begin{aligned} H \ =& 2\big[\kappa_{1,2}(E_2E_1^*) + \kappa_{1,3}(E_3^*E_1) + \kappa_{2,1}(E_1E_2^*) \\ &+\kappa_{2,3}(E_3^*E_2) + \kappa_{3,1}(E_1E_3^*) + \kappa_{3,2}(E_2^*E_3)\big], \end{aligned} \tag{4}$$

Eq. (1) could be re-organized as the following matrix form

$$\boldsymbol{E}'(z) + \boldsymbol{M} \cdot \boldsymbol{E}(z) = 0, \tag{5}$$

matrix M in Eq. (6) could be expressed as

$$\boldsymbol{M} = \begin{bmatrix} 0 & -\kappa_{1,2} & -\kappa_{1,3} \\ -\kappa_{2,1} & 0 & -\kappa_{2,3} \\ -\kappa_{3,1} & -\kappa_{3,2} & 0 \end{bmatrix}. \tag{6}$$

The superscript T in vector $\boldsymbol{E} = (E_1, E_2, E_3)^T$ means the transpose operation. For the cases of the uniform waveguide array, the mutual coupling coefficients are the same. The matrix (6) is tridiagonal and symmetric ($M^T = M$). It is of interest to find the eigenmodes of Eq. (1). Assume the amplitude of mode l^{th} in waveguide j^{th} to be

$$\boldsymbol{E}^{(l)}(z) = \boldsymbol{x}^{(l)} \cdot \exp(i\lambda_l z). \tag{7}$$

Then the resulting eigenvalue problem becomes

$$\boldsymbol{M} \cdot \boldsymbol{x}^{(l)} = -i\lambda_l \boldsymbol{x}^{(l)}. \tag{8}$$

*Supported by the National Basic Research Programme of China under Grant No 2007CB613405, and the National Natural Science Foundation of China under Grant No 60777015.

**Present address: Department of Electrical and Computer Engineering, University of Virginia, UAS. Email: zl4s@virginia.edu
***To whom correspondence should be addressed. Email: iseejxq@zju.edu.cn

© 2008 Chinese Physical Society and IOP Publishing Ltd

It is clear that the eigenvalue λ_l denotes the propagation constant of mode l^{th}, and the eigenvector $\boldsymbol{x}^{(l)}$ denotes the normalized amplitude distribution of optical field in transversal direction or x direction in Fig. 1.

By using Laplace transform and LU decomposition (or triangular matrix decomposition),[8] a more detailed analytic evolution equation for arbitrary optical input could be given. To achieve an accurate simulation, coupling between non-neighbourhood waveguide j^{th} and $j+2^{th}$ is taken into account besides neighbourhood coupling. For the 2-D cases, while the size and material of two waveguides are the same, the coupling coefficient could be approximated as[10]

$$\kappa = \frac{2q^2h^2}{\beta(wq+2)(h^2+q^2)}\exp(-qs), \tag{9}$$

where $q^2 = \beta^2 - n_i^2k^2$, $h^2 = n_f^2k^2 - \beta^2$, $k = 2\pi/\lambda$, n_i denotes the refractive index of the cladding, and n_f denotes the refractive index of the waveguides, β is the propagation constant of each waveguide, w is the width of each waveguide, s is the distance between these two waveguides.

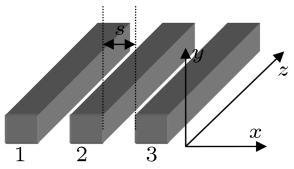


Fig. 1. Finite waveguide arrays.

From Eq. (9), coupling coefficient for any two waveguides with the same distance is the same. Therefore we rewrite the matrix (7) as

$$\boldsymbol{M} = \begin{bmatrix} 0 & -\kappa_1 & -\kappa_2 \\ -\kappa_1 & 0 & -\kappa_1 \\ -\kappa_2 & -\kappa_1 & 0 \end{bmatrix}, \tag{10}$$

where κ_1 is the coupling coefficient between neighbourhood waveguides and κ_2 is the coupling coefficient between non-neighbourhood waveguides j^{th} and $j+2^{th}$; κ_1 is always larger than κ_2 in magnitude which is attributed to a shorter distance s according to Eq. (9). Performing the Laplace transform to the symbol z in Eq. (5), then we have

$$A \cdot \bar{\boldsymbol{E}}(p) = \boldsymbol{E}(0), \tag{11}$$

$$A = \begin{bmatrix} p & -i\kappa_1 & -i\kappa_2 \\ -i\kappa_1 & p & -i\kappa_1 \\ -i\kappa_2 & -i\kappa_1 & p \end{bmatrix}, \tag{12}$$

$\boldsymbol{E}(0) = (E_1(0), E_2(0), E_3(0))^T$, in which the complex number $E_m(0)$ denotes the input optical field at the m^{th} waveguide. $\bar{\boldsymbol{E}}(p) = (\bar{E}_1(p), \bar{E}_2(p), \bar{E}_3(p))^T$ and $\bar{E}_m(p)$ denote the Laplace transform of the optical evolution equation in the m^{th} waveguide. According to LU decomposition, we can decompose matrix (10) into the following form:

$$A = L \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}. \tag{13}$$

With Eq. (13), Eq. (11) reduces to

$$L \cdot y = \boldsymbol{E}(0), \quad U \cdot \bar{\boldsymbol{E}}(p) = y. \tag{14}$$

By making use of the chase method,[8] it is easy to fix each element of matrix L and U. After that, y can be solved from $L \cdot y = \boldsymbol{E}(0)$ and then $\bar{\boldsymbol{E}}(p)$ from $U \cdot \bar{\boldsymbol{E}}(p) = y$. Evolution equations are obtained from the inversion of Laplace transform $\bar{\boldsymbol{E}}(p)$. Actually this LU decomposition method and its algorithm still hold true when the coupling between any two waveguides is considered. The only difference is that matrix A will have more non-zero element.

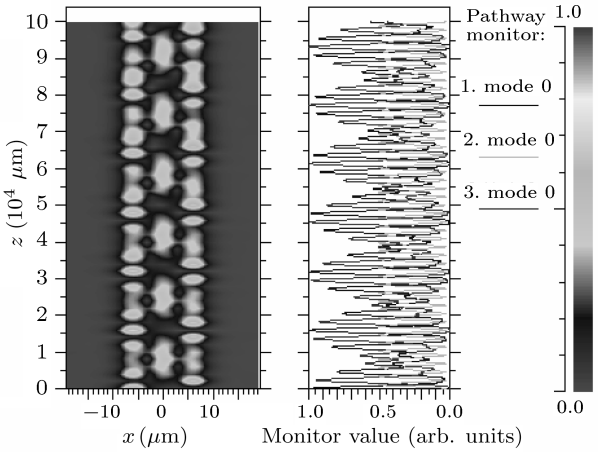


Fig. 2. Field evolution at a waveguide width of 4 μm with light incidence condition $\boldsymbol{E}(0) = (1, 0, 0)^T$.

For three waveguide arrays, the solution from LU decomposition method shows

$$\lambda_1 = -\kappa_2, \quad \lambda_{2,3} = \lambda_\pm = \frac{1}{2}\Big(\kappa_2 \pm \sqrt{\kappa_2^2 + 8\kappa_1^2}\Big). \tag{15}$$

If $\kappa_2 = 0$, considering only the neighbourhood coupling κ_1, Eq. (15) is degenerated into

$$\lambda_1' = 0, \quad \lambda_{2,3}' = \lambda_\pm' = \pm\sqrt{2}\kappa_1. \tag{16}$$

This is the result of Ref. [7]. Since $\lambda_1' = 0$, there is no revival period for the first mode. While $\lambda_+' = \lambda_-'$, the revival periods for the second and third modes are the same. Therefore from Eq. (16), one would conclude that such a system always possesses Talbot revival.[7] However, when the width or distance is small, non-neighbourhood coupling should be considered. Equation (15) should be employed to calculate eigenvalues. Since a square root $\sqrt{\kappa_2^2 + 8\kappa_1^2}$ is extracted, it is quite possible that λ_+ and λ_- are irrational and the ratio λ_+/λ_- is also irrational. If so, it is impossible for the

second and third modes to achieve revival at the same propagation length and Talbot effect does not exist. Thus, in a word, Talbot revival does not always show up in an array with three waveguides. This is totally different from the conclusion in Ref. [7].

Suppose the input field $E(0) = (1,0,0)^T$, which means that one-unit optical input is for the first waveguide while the other two waveguides have no light at the entrances at all. The solution from Matlab shows the optical field in three waveguides centre as follows:

$$E(z) = \begin{cases} \frac{1}{2}\exp(-i\kappa_2 z) - \frac{1}{4}\Big(\frac{\kappa_2}{\sqrt{\kappa_2^2+8\kappa_1^2}} - 1\Big)\exp\Big[\frac{1}{2}(\kappa_2 - \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big] + \frac{1}{4}\Big(\frac{\kappa_2}{\sqrt{\kappa_2^2+8\kappa_1^2}} + 1\Big) \\ \cdot \exp\Big[\frac{1}{2}(\kappa_2 + \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big], \quad \text{for waveguide 1,} \\ \frac{\kappa_1}{\sqrt{\kappa_2^2+8\kappa_1^2}}\exp\Big[\frac{1}{2}(\kappa_2 + \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big] - \frac{\kappa_1}{\sqrt{\kappa_2^2+8\kappa_1^2}}\exp\Big[\frac{1}{2}(\kappa_2 - \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big], \\ \quad \text{for waveguide 2,} \\ \frac{1}{2}\exp(-i\kappa_2 z) - \frac{1}{4}\Big(\frac{\kappa_2}{\sqrt{\kappa_2^2+8\kappa_1^2}} - 1\Big)\exp\Big[\frac{1}{2}(\kappa_2 - \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big] + \frac{1}{4}\Big(\frac{\kappa_2}{\sqrt{\kappa_2^2+8\kappa_1^2}} + 1\Big) \\ \cdot \exp\Big[\frac{1}{2}(\kappa_2 + \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big], \quad \text{for waveguide 3.} \end{cases} \tag{17}$$

When the input field is $\boldsymbol{E}(0) = (0,1,0)^T$, the evolution becomes

$$E(z) = \begin{cases} \frac{\kappa_1}{\sqrt{\kappa_2^2+8\kappa_1^2}}\exp\Big[\frac{1}{2}(\kappa_2 + \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big] - \frac{\kappa_1}{\sqrt{\kappa_2^2+8\kappa_1^2}}\exp\Big[\frac{1}{2}(\kappa_2 - \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big], \\ \quad \text{for waveguide 1,} \\ \frac{1}{2}\Big(\frac{\kappa_2}{\sqrt{\kappa_2^2+8\kappa_1^2}} + 1\Big)\exp\Big[\frac{1}{2}(\kappa_2 - \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big] - \frac{1}{2}\Big(\frac{\kappa_2}{\sqrt{\kappa_2^2+8\kappa_1^2}} - 1\Big) \\ \cdot \exp\Big[\frac{1}{2}(\kappa_2 + \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big], \quad \text{for waveguide 2,} \\ \frac{\kappa_1}{\sqrt{\kappa_2^2+8\kappa_1^2}}\exp\Big[\frac{1}{2}(\kappa_2 + \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big] - \frac{\kappa_1}{\sqrt{\kappa_2^2+8\kappa_1^2}}\exp\Big[\frac{1}{2}(\kappa_2 - \sqrt{\kappa_2^2+8\kappa_1^2})iz\Big], \\ \quad \text{for waveguide 3.} \end{cases} \tag{18}$$

Due to the symmetry in the geometry, the evolution for input $\boldsymbol{E}(0) = (0,0,1)^T$ is similar to the one for $\boldsymbol{E}(0) = (1,0,0)^T$, while the evolution equations in the first and third waveguides should exchange with each other.

Anyway, what is the revival like when a three-waveguide system is involved? We resort to BMP simulation to help us test the above conclusion. Two dielectric slab waveguides are considered with step-profile refraction indices of buffer layer $n_i = 3.337$ and core layer $n_f = 3.380$. The waveguide width is $4\,\mu$m and waveguide spacing is $2\,\mu$m. The propagating wave has a wavelength of $1.15\,\mu$m. Waveguides are numbered from left to right. Let light beam go into waveguide 1. Let us investigate Fig. 2, which is the result of BPM simulation. The curves show that the amplitude distributions of all the modes in each wave package are different, which indicates a lack of common revival distance. The wave pattern in the left picture also shows little sign of self-imaging after a propagation of 100 mm. Clearly, there is no Talbot revival in this case. However, if we change the parameters slightly, the situation will be totally different. Let the waveguide width be $6\,\mu$m while keep the spacing $2\,\mu$m to weaken the coupling, and then run the program again.

Since the revival length becomes longer, we extend the simulation length to 200 mm to better examine the result. The curves in Fig. 3 shows perfect reappearance of field amplitude in each wave package and the wave pattern also shows good similarity during the propagation length. The reappearance phenomenon in this case could be classified as Talbot revival. Therefore, array of three waveguides does not always possess Talbot effect.

For finite waveguide array to achieve Talbot revival, generally it would require the ratio between eigenvalues λ_i/λ_j be rational for any $i, j,\ i \neq j$. If those ratios are all rational, we could expect to find a common propagation length, for which all modes will experience phase differences of multiples of 2π and reconstruct the original field pattern, i.e. the self-

imaging phenomenon (or Talbot effect).

To achieve Talbot revival in a three waveguide system, first $\lambda_{\pm}$ should satisfy:

$$\lambda_+/\lambda_- = m/v, \quad m \in Z^+, \quad v \in Z^-, \qquad (19)$$

which results in

$$mv\kappa_2^2 + 2(m+v)^2\kappa_1^2 = 0, \quad m \in Z^+, \quad v \in Z^-, \quad (20)$$

where Z^+/Z^- is a positive/ negative integer. Transformation of Eq. (19) also results in

$$\kappa_2/\sqrt{\kappa_2^2 + 8\kappa_1^2} = (m+v)/(m-v), \qquad (21)$$

which indicates that the ratios between λ_1 and $\lambda_{2,3}$ are also rational. Therefore Eq. (19) could serve as a sufficient condition for Talbot effect in a three-waveguide system.

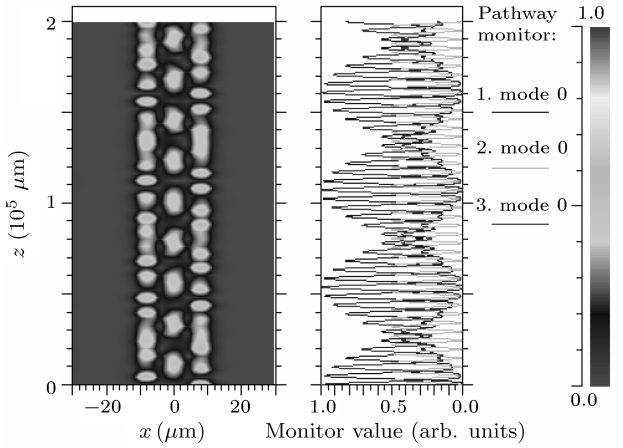


Fig. 3. Field evolution at waveguide width of 6 μm with light incidence condition $\boldsymbol{E}(0) = (1, 0, 0)^T$.

When input field is symmetric, two modes with$\lambda_{\pm} = \frac{1}{2}\left(\kappa_2 \pm \sqrt{\kappa_2^2 + 8\kappa_1^2}\right)$ appear. Equation (20) is the condition for Talbot effect in three-waveguide system with symmetric input. In fact, Talbot effect is not only determined by structure parameters such as κ_1 and κ_2, but also determined by the light incidence condition. Again keep the waveguide width as 4 μm and the waveguide spacing as 2 μm, but let light beam exited from the middle waveguide, or the waveguide 2. The BPM simulation result is shown in Fig. 4. Obviously, there is Talbot revival in the device at this time. When input field is symmetric, only one mode with $\lambda = -\kappa_2$ appears. The Talbot effect exists.

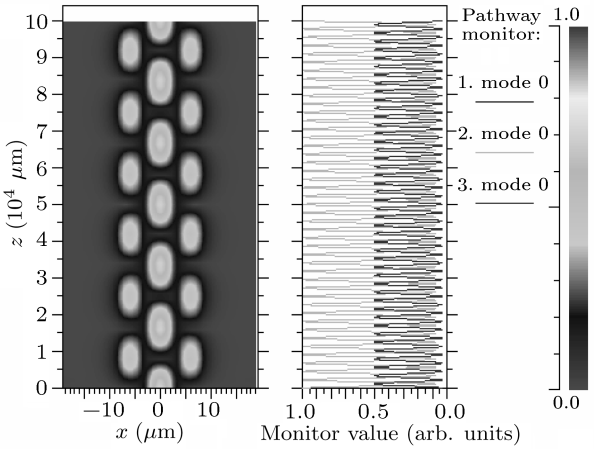


Fig. 4. Field evolution at waveguide width of 4 μm with light incidence condition [0,1,0].

Talbot effects in three waveguide arrays are closely related to coupling parameters and incidence condition. When coupling between non-neighbourhood waveguides is considered, the three-waveguide array does not always possess Talbot effect. The sufficient conditions to achieve Talbot effect are discussed. This methods are extended analogously in finite ($N > 3$) waveguide arrays.

References

[1] Talbot H F 1836 *Philos. Mag.* **9** 401
[2] Bryngdahl O 1973 *J. Opt. Soc. Am.* **63** 416
[3] Packross B, Eschbach R and Bryngdahl O 1986 *Opt. Commun.* **56** 394
[4] Leger J R and Swanson GJ 1990 *Opt. Lett.* 15 288
[5] Soldano L B and Pennings E C 1995 *J. Lightwave Technol.* **13** 615
[6] Robert I, Daniel A M, Demetrios N C and George I S 2005 *Phys. Rev. Lett.* **95** 053902
[7] Nikolaos K E and Demetrios N C 2005 *Opt. Commun.* **246** 345
[8] Compiling Group of Mathematics Manual 2004 *Mathematics Manual* (Beijing, Higher Education Press) chap 4 (in Chinese)
[9] Marcuse D 1991 *Theory of Dielectric Optical Waveguides* (San Diego: Academic) chap 5
[10] Hunsperger R G 1984 *Integrated Optics: Theory and Technology* 2nd edn (Berlin: Springer) chap 8 p 141