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Soliton Structure of a Higher Order (2+1)-Dimensional Nonlinear Evolution Equation of Barothropic Relaxing Media beneath High-Frequency Perturbations

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From the dynamical equation of barothopic relaxing media beneath pressure perturbations, followed with the reductive perturbative analysis, we derive and investigate the soliton structure of a (2+1)-dimensional nonlinear evolution equation describing high-frequency regime of perturbations. Thus, by means of the Hirota's bilinearization method, we unearth three typical patterns of loop-, cusp- and hump-like shapes depending strongly upon a dissipation parameter.

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In real natural world, there exist very complicated phenomena closely related to nonlinear systems. However, in various cases, the real natural phenomena are too intricate to be described only by virtue of (1+1)-dimensional nonlinear partial differential evolution (NLPDE) equations. Thus, a great deal of interest has recently been paid to higher-dimensional NLPDE equations, especially, the (2+1)-dimensional cases ([1] and references therein). Such higher-dimensional systems have been shown to possess soliton solutions which arise as a balance between nonlinearity and dispersion. One underlying query arising consists of the survival of the soliton properties of solutions to such systems with higher order nonlinearity. In the wake of such query, we consider a barothropic relaxing medium $p = p(\rho, \lambda)$ under high-frequency perturbations $p' \ll p_0$. The quantities p and ρ stand for pressure and mass density, respectively. The constant λ is an additional parameter. The pressure p_0 is measured at the unperturbed state. For $\lambda = 1$, $p = p_f$, inner interaction processes are frozen and the velocities for fast processes are defined as $v_f^2 = dp_f/d\rho$. For $\lambda = 0$, $p = p_e$, there is local thermodynamic equilibrium and the velocities for slow processes are defined as $v_e^2 = dp_e/d\rho$.

A few years ago, Danylenko *et al.*[2] have proposed dynamical equations of state for multicomponent relaxing media. In the wake of such result, after a suitable expansion of the specific volume $V \equiv \rho^{-1}$ as power series of perturbation p' with accuracy $o(p'^3)$, the following dynamical equation is written as

$$\tau\partial_t\Big(\partial^2_{x_m x^m}p' - \frac{1}{v_f^2}\partial_t^2 p' + \alpha_f\partial_t^2 p'^2 + a_f\partial_t^2 p'^3\Big) + \partial^2_{x_m x^m}p' - \frac{1}{v_e^2}\partial_t^2 p' + \alpha_e\partial_t^2 p'^2 + a_e\partial_t^2 p'^3 = 0, \quad (1)$$

where $m = 1, \ldots, N$ and τ is the relaxation time; constants α_e and α_f represent positive-valued second order expansion coefficients of V_e and V_f, respectively; constants a_e and a_f stand for positive-valued third order expansion coefficients of V_e and V_f, respectively; quantity $x^m \equiv (x, y, \ldots)$ stands for position-vector and $\partial^2_{x_m x^m} = \partial^2_{x_1 x^1} + \partial^2_{x_2 x^2} + \partial^2_{x_3 x^3} + \cdots$, $(m = 1, \ldots, N)$. It is noted that any repeated index refers to summation with respect to Einstein's notation. We assume the notation $\partial_t^2 \equiv \partial_{tt}^2 = \partial^2/\partial t^2$. In order to investigate the Eq. (1), the multiscale method[3] is used. Thus, defining the quantity $\epsilon = \tau\omega$ (ω being the frequency of the processes) chosen to be small (large) parameter, after introducing the following independent variables $T_0 = t\omega$, $T_{-4} = t\omega\epsilon^{-4}$, $X_0^m = x^m\omega$, $X_{-4}^m = x^m\omega\epsilon^{-4}$, $(m = 1, \ldots, N)$ into Eq. (1), seven coupled equations are derived. From this coupled system, there are two leading equations expressed in terms of T_0 and X_0^m $(m = 1, \ldots, N)$, describing low-frequency perturbations, and the two other expressed in terms of T_{-4} and X_{-4}^m $(m = 1, \ldots, N)$ describing high-frequency perturbations. These two regimes of perturbation are worthy to study owing to interesting fauna of features that are hidden. For example, in a recent study, Vakhnenko[4] has shown that the low-frequency regime of pressure perturbations in a (1+1)-dimensional manifold is modelled by the Korteweg–de Vries–Burgers equation.

Nonetheless, in the present work, we focus our interests to high-frequency perturbations and we derive the following evolution equation:

$$(\partial_{x_n} + \partial_{x_m})(\partial_{x_n} + \partial_{x_m})\big(p' + \alpha_f v_f^2 p'^2 + a_f v_f^2 p'^3\big) - v_f^{-2}\partial_t^2 p' + \beta_f\partial_{x_m}J^m + \gamma_f p' = 0, \ (n < m), \quad (2)$$

where $J^m = p'$ $(m = 1, \ldots, N)$, and the quantities β_f and γ_f are expressed as

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$$\beta_f = \frac{v_f^2 - v_e^2}{\tau v_e^2 v_f}, \quad \gamma_f = \frac{v_f^4 - v_e^4}{2\tau^2 v_e^4 v_f^2}. \tag{3}$$

Equation (2) is obtained in the following way. A dispersion relation for the linearized Eq. (2) is written down in the form $v_f^{-2}\omega^2 = (k_n + k_m)(k_n + k_m) + j\beta_f \sum_{m=1}^{N} k_m - \gamma_f$ with $(n < m = 1, \ldots, N)$. The nonlinear terms is reconstructed in agreement with the initial equation. We note here that Eq. (2) has dissipative β_f-terms and dispersive γ_f-terms. If $\alpha_f = a_f = \beta_f = 0$, Eq. (2) reduces to a typical modified Klein–Gordon equation in (N+1)-dimensional space. We particularly focus our interests to $N = 2$ case. Then, it comes

$$(\partial_x + \partial_y)^2 p' - v_f^{-2}\partial_t^2 p' + \alpha_f v_f^2 (\partial_x + \partial_y)^2 p'^2 + a_f v_f^2 (\partial_x + \partial_y)^2 p'^3 + \beta_f (\partial_x + \partial_y) p' + \gamma_f p' = 0. \tag{4}$$

If $a_f = 0$ and the y-terms are removed, Eq. (4) is reduced to the (1+1)-dimensional Vakhnenko (V) equation.[4] In the high-frequency perturbations regime, it is worth considering the following accuracy:

$$(\partial_x + \partial_y)^2 - v_f^{-2}\partial_t^2 \approx 2(\partial_x + \partial_y)(\partial_x + \partial_y + v_f^{-1}\partial_t), \tag{5}$$

to further investigate Eq. (4). It is actually clear that there exist many techniques to find the soliton solutions to some NLPDE equations. For example, using the 'multilinear variable separation approach' recently investigated by Tang and Lou,[1] leads to many interesting solutions. Applying the extended painlevé truncation ([5] and references therein) to the current system is full of interests. Thus, for a given NLPDE equation, there exist many solutions. The Hirota bilinear method,[6] even though looks standard, has been shown to be powerful. This method is useful to find nontrivial solutions to many NLPDE equations.[7,8]

It is important to note that Eq. (4) has interesting aspects. Generally, in order to investigate a system, it is useful to transform this system to another system without any dimensional units. Owing to the plenty exotic structures hidden in Eq. (4), we shall regard one important aspect by considering $\alpha_f = 0$. In this case, Eq. (4) is transformed to

$$(\partial_{\tilde{x}} + \partial_{\tilde{y}})\Big[\partial_{\tilde{t}} - \frac{1}{2}\tilde{u}^2(\partial_{\tilde{x}} + \partial_{\tilde{y}})\Big]\tilde{u} + \tilde{\alpha}(\partial_{\tilde{x}} + \partial_{\tilde{y}})\tilde{u} - \tilde{u} = 0, \tag{6}$$

provided the dimensionless quantities $\tilde{x}$, $\tilde{y}$, $\tilde{t}$, $\tilde{u}$ and $\tilde{\alpha}$ expressed as follows:

$$\tilde{x} = -\sqrt{\frac{\gamma_f}{6}}(x - v_f t), \quad \tilde{y} = -\sqrt{\frac{\gamma_f}{6}}(y - v_f t),$$
$$\tilde{t} = v_f\sqrt{\frac{3\gamma_f}{2}}t, \quad \tilde{u} = v_f\sqrt{a_f}p', \quad \tilde{\alpha} = \frac{\beta_f}{\sqrt{6\gamma_f}}, \tag{7}$$

hold. The $\tilde{\alpha}$-term stands for the 'dissipative' term as introduced and defined by Vakhnenko[4] who regards the 'dissipation' in terms of partial derivatives with respect to space coordinate. This is particularly novel since as usual, dissipation is defined in terms of partial derivatives with respect to time. The dissipation as defined by Vakhnenko[4] is particularly underlying since it leads to stable structures with conserved energy. In addition, this kind of dissipation is useful in surveying some kind of structures with generation of tail often encountered in nonlinear dispersive media with weak dissipation.[9]

Indeed, it is possible to reduce Eq. (6) to a (1+1)-dimensional system up to the transformation $\tilde{z} = (\tilde{x} + \tilde{y})/2$. However, it seems not simple to express the solutions in terms of the independent variables $\tilde{x}$ and $\tilde{y}$. Thus, performing variable transformation, we introduce new independent variables X, T_1 and T_2 as follows:

$$\tilde{x} = T_1 - \frac{1}{2}\int_{-\infty}^{X} U^2 dX' + \tilde{x}_0,$$
$$\tilde{y} = T_2 - \frac{1}{2}\int_{-\infty}^{X} U^2 dX' + \tilde{y}_0,\ \tilde{t} = X, \tag{8}$$

where $\tilde{x}_0$ and $\tilde{y}_0$ stand for arbitrary constants and $\tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) = U(T_1, T_2, X)$. Eq. (6) is rewritten as

$$U_{XT_1} + U_{XT_2} + \tilde{\alpha}(U_{T_1} + U_{T_2}) - (1 + \varphi + \phi)U = 0, \tag{9}$$

where $\varphi = -\int_{-\infty}^{X} UU_{T_1} dX'$ and $\phi = -\int_{-\infty}^{X} UU_{T_2} dX'$. Subscripts with respect to X, T_1 and T_2 denote partial differentiations. Thus, whilst using the ansatzes $\varphi = Z_{1T_1} - 1$ and $\phi = Z_{2T_2} - 1$, Eq. (6) bilinearizes to

$$(D_X D_{T_1} + D_X D_{T_2} + \tilde{\alpha}(D_{T_1} + D_{T_2}) - 1)G \cdot F = 0,$$
$$D_X^2 F \cdot F = G^2/2, \tag{10}$$

provided

$$U = \frac{G}{F},\ Z_1 = T_1 - 2(\ln(F))_X,$$
$$Z_2 = T_2 - 2(\ln(F))_X, \tag{11}$$

hold. Notations D_X, D_{T_1} and D_{T_2} denote Hirota operators.[6] Expanding F and G in a suitable formal power series, a one-soliton solution to Eq. (9) is given by

$$U = 2K\,\mathrm{sech}(\theta), \quad Z_1 = T_1 - 2K[\tanh(\theta) + 1],$$
$$Z_2 = T_2 - 2K[\tanh(\theta) + 1], \tag{12}$$

where $\theta = KX - \omega_1 T_1 - \omega_2 T_2 + \theta_0$, θ_0 being an arbitrary constant. The dispersion relation is given by

$$(K + \tilde{\alpha})(\omega_1 + \omega_2) + 1 = 0. \tag{13}$$

As a result, assuming that $\omega_1 = Kv_1$ and $\omega_2 = Kv_2$ such that $v = v_1 + v_2 < 0$, we find out three typical pattern formations. Indeed,

(1) for $\tilde{\alpha} < \sqrt{1/2|v|}$, loop-like pattern is obtained (see Fig. 1 in the case of $v = -0.24$);

(2) for $\tilde{\alpha} = \tilde{\alpha}_c = \sqrt{1/2|v|}$, cusp-like pattern is obtained (see Fig. 2 in the case of $v = -0.24$ where $\tilde{\alpha}_c = 1,44$);

(3) finally, for $\tilde{\alpha} > \sqrt{1/2|v|}$, hump-like pattern is obtained (see Fig. 3 in the case of $v = -0.24$).

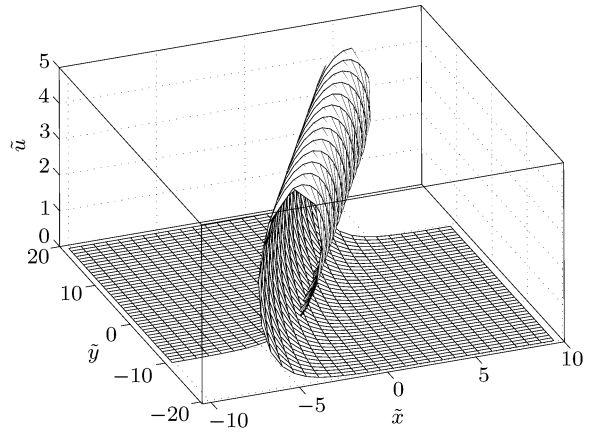


Fig. 1. Loop-like pattern formation $\tilde{u}(\tilde{x}, \tilde{y})$ at initial time $\tilde{t} = 0$: dissipation parameter $\tilde{\alpha} < 1,44$ and velocity $v = -0.24$.

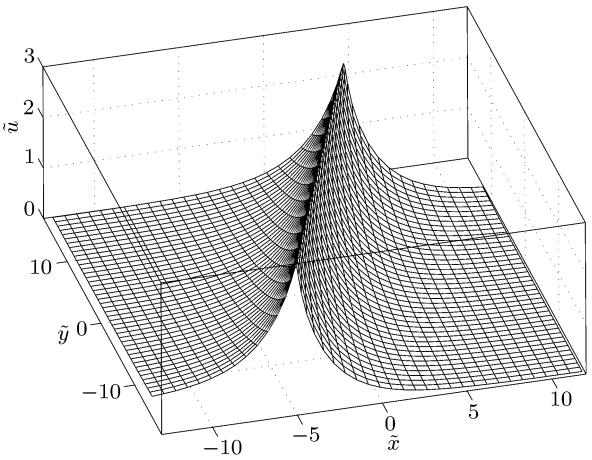


Fig. 2. Cusp-like pattern formation $\tilde{u}(\tilde{x}, \tilde{y})$ at initial time $\tilde{t} = 0$: dissipation parameter $\tilde{\alpha} = 1,44$ and velocity $v = -0.24$.

These illustrative curves are plotted at initial time $\tilde{t} = 0$. In summary, we have investigated a (2+1)-dimensional NLPDE equation (see Eq. (6)) with the soliton structure. This (2+1)-dimensional NLPDE equation is regarded as a (2+1)-dimensional version of the (1+1)-dimensional Schäfer-Wayne short pulse (SWSP) equation.[10] This equation actually deserves further scientific interests both from the viewpoint of the investigation of the propagation of high-frequency perturbations and from the viewpoint of the existence of stable wave formations. It is actually true that the reductive perturbation method and the Hirota bilinear method seem to be standard. Indeed, the present study aims to point out the properties of the system under investigation when higher dimensional space and higher order nonlinearity are considered. It does not specialize on a method or technique of investigation. As mentioned above, there exist many other techniques to tackle the problem.

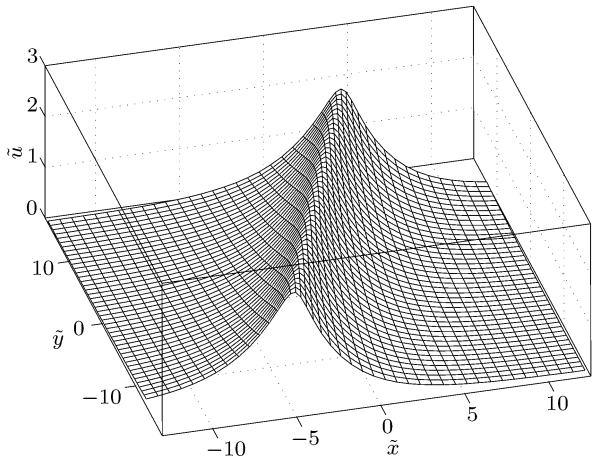


Fig. 3. Hump-like pattern formation $\tilde{u}(\tilde{x}, \tilde{y})$ at initial time $\tilde{t} = 0$: dissipation parameter $\tilde{\alpha} > 1,44$ and velocity $v = -0.24$.

Moreover, an essential remark on Eq. (6) should be noted here concerning rotating solitons. If the physical quantity $\tilde{u}$ is treated as a complex-valued observable, it is understandable if the considered media are not homogeneous or merely isotropic for example, then, from Eq. (6), we construct an equation as follows:

$$\begin{aligned}&(\partial_{\tilde{x}} + \partial_{\tilde{y}})\Big[\partial_{\tilde{t}} - \frac{1}{2}\tilde{q}\tilde{q}^{\star}(\partial_{\tilde{x}} + \partial_{\tilde{y}})\Big]\tilde{q}\\ &+ \tilde{\alpha}(\partial_{\tilde{x}} + \partial_{\tilde{y}})\tilde{q} - \tilde{q} = 0,\end{aligned} \tag{14}$$

where $q = q^r + \imath q^{im}$ stands for a complex-valued observable, symbol $(\star)$ refers to complex conjugation and $\imath^2 = -1$. Thus, one derive the following set

$$\begin{aligned}&Q^r_{XT_1} + Q^r_{XT_2} + \tilde{\alpha}(Q^r_{T_1} + Q^r_{T_2})\\ &- (1 + Z_{1T_1} + Z_{2T_2})Q^r = 0,\\ &Q^{im}_{XT_1} + Q^{im}_{XT_2} + \tilde{\alpha}(Q^{im}_{T_1} + Q^{im}_{T_2})\\ &- (1 + Z_{1T_1} + Z_{2T_2})Q^{im} = 0,\\ &Z_{1XT_1} = -(Q^r Q^r_{T_1} + Q^{im} Q^{im}_{T_1}),\\ &Z_{1XT_2} = -(Q^r Q^r_{T_2} + Q^{im} Q^{im}_{T_2}),\end{aligned} \tag{15}$$

where $\tilde{q}(\tilde{x}, \tilde{y}, \tilde{t}) = Q(T_1, T_2, X)$ such that independent variables $\tilde{x}$, $\tilde{y}$ and $\tilde{t}$ now stand for

$$\begin{aligned}\tilde{x} &= T_1 - \frac{1}{2}\int_{-\infty}^{X} QQ^{\star} dX' + \tilde{x}_0,\\ \tilde{y} &= T_2 - \frac{1}{2}\int_{-\infty}^{X} QQ^{\star} dX' + \tilde{y}_0,\ \tilde{t} = X.\end{aligned} \tag{16}$$

It comes that Eq. (15) is closely related to the coupled dispersionless systems[11−14] recently investigated by

Kakuhata and Konno.[8] In addition, Eq. (15) is also observed as a coupled (2+1)-dimensional version of the complex-form of the Schäfer–Wayne short pulse (SWSP) equation[10] that has been subject to many recent investigations.[15−20] A one-soliton solution to Eq. (15) is written as follows:

$$\begin{aligned} Q &= A\exp(\theta^{im})\mathrm{sech}(\theta^r), \\ Z_1 &= T_1 - 2K^r[\tanh(\theta^r)+1], \\ Z_2 &= T_2 - 2K^r[\tanh(\theta^r)+1], \end{aligned} \tag{17}$$

where $\theta = \theta^r + \imath\theta^{im} = KX - \omega_1 T_1 - \omega_2 T_2 + \theta_0$, θ_0 being an arbitrary complex-valued constant, $K = K^r + \imath K^{im}$ and $\omega_j = \omega_j^r + \imath\omega_j^{im}$ $(j = 1, 2)$. The amplitude A is given by $A = 2K^r$. The corresponding dispersion relation is given by Eq. (13). As a result, one easily finds that solutions given by Eq. (17) in terms of Q^r and Q^{im}, possess a nonzero angular momentum, i.e. the previous patterns depicted above rotate with an angular frequency $\Omega = \omega_1^{im} + \omega_2^{im}$. Specifically, if we assume that $\omega_1^r = v_1 K^r$, $\omega_2^r = v_2 K^r$, $K^{im} = \zeta(\omega_1^{im} + \omega_2^{im})$ and $v = v_1 + v_2 < 0$, K^r and Ω may be expressed as follows

$$K^r = \frac{\tilde{\alpha}}{\zeta|v| - 1}, \quad \Omega = \frac{1}{\sqrt{\zeta}}\sqrt{1 - \frac{\zeta\tilde{\alpha}^2 v^2}{(\zeta|v|-1)^2}}, \tag{18}$$

provided $\dfrac{1}{|v|} < \zeta < \dfrac{2}{|v|}$. Thus, setting $\tilde{\alpha}_c = \dfrac{\zeta|v|-1}{\sqrt{2|v|}}$ and $\tilde{\alpha}_s = \dfrac{\zeta|v|-1}{|v|\sqrt{\zeta}}$, it comes that,

(1) for $\tilde{\alpha} \geq \tilde{\alpha}_s$, no rotating pattern formation is expected;

(2) for $\tilde{\alpha}_c < \tilde{\alpha} < \tilde{\alpha}_s$, rotating loop-like pattern formation is derived (see Fig. 1 for $v = -0.24$ and $\zeta = 13/3$ at initial time $\tilde{t} = 0$);

(3) for $\tilde{\alpha} = \tilde{\alpha}_c$, rotating cusp-like pattern formation is derived (see Fig. 2 for $v = -0.24$ and $\zeta = 13/3$ at initial time $\tilde{t} = 0$; the parameter $\tilde{\alpha}_c = 0,057$);

(4) finally, for $\tilde{\alpha} < \tilde{\alpha}_c$, rotating hump-like pattern formation is derived (see Fig. 3 for $v = -0.24$ and $\zeta = 13/3$ at initial time $\tilde{t} = 0$).

In the nature, relaxing media are so widely found. For instance, one can find gas-media, steam-liquid media, electrons-gas in conductors, just to name a few. Since conservative flow equations have been used to describe the behaviour of such media, the suitable fields for applications of the previous results are found in soliton theory,[7] hydrodynamics,[21] string physics,[8] just to name a few.

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