CHIN.PHYS.LETT. Vol. 25, No. 9 (2008) 3169

Hyperchaos–chaos–Hyperchaos Transition in a Class of On–Off Intermittent Systems Driven by a Family of Generalized Lorenz Systems *

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(Received 19 May 2008)

Blowout bifurcation in nonlinear systems occurs when a chaotic attractor lying in some symmetric subspace becomes transversely unstable. A class of five-dimensional continuous autonomous systems is considered, in which a two-dimensional subsystem is driven by a family of generalized Lorenz systems. The systems have some common dynamical characters. As the coupling parameter changes, blowout bifurcations occur in these systems and brings on change of the systems' dynamics. After the bifurcation the phenomenon of on–off intermittency appears. It is observed that the systems undergo a symmetric hyperchaos–chaos–hyperchaos transition via or after blowout bifurcations. An example of the systems is given, in which the drive system is the Chen system. We investigate the dynamical behaviour before and after the blowout bifurcation in the systems and make an analysis of the transition process. It is shown that in such coupled chaotic continuous systems, blowout bifurcation leads to a transition from chaos to hyperchaos for the whole systems, which provides a route to hyperchaos.

PACS: 05. 45. Jn, 05. 45. Pq

Recently, there has been an increasing interest in the study of deterministic higher-dimensional nonlinear systems, and in particular some coupled systems.[1−18] When varying a system parameter, such systems can produce some interesting complex phenomena such as synchronization,[1−5] symmetry-breaking[6] or symmetry-increasing,[7] on–off intermittency[1,6−9] shown to be related to blowout bifurcation [8−11] and transition from chaos to hyperchaos.[3−5,7,10,13−18] In fact, some hyperchaotic systems proposed are designed by nonlinearly coupling two low-dimensional systems.[19,20]

Hyperchaos was first presented by Rössler in Ref. [21] to name chaotic phenomenon with more than one positive Lyapunov exponent. There have been reports that in some coupled identical chaotic systems chaos–hyperchaos transition occurs through blowout bifurcation.[3−5,16−18] Kim *et al.*[17] investigated the dynamical origin for the occurrence of asynchronous hyperchaos and chaos via blowout bifurcation in coupled identical chaotic systems. The theory of unstable periodic orbits (UPOs) has been used to explain higher-dimensional dynamical phenomenon of blowout bifurcation[22] and chaos–hyperchaos transition.[23]

These reports of chaos–hyperchaos transition via blowout bifurcation are generally in the synchronization of chaotic system. In our previous work,[10] we also find that when using a modified Lorenz chaotic system to drive a two-dimensional nonlinear system, the whole coupled system undergoes a symmetric hyperchaos–chaos–hyperchaos transition via blowout bifurcations. In this study, we consider a class of five-dimensional nonlinear systems driven by a family of generalized Lorenz systems. As the coupling parameter changes, we observe that the systems undergo a symmetric hyperchaos–chaos–hyperchaos transition via or after blowout bifurcations. It is found that blowout bifurcation leads to the transition. Our work indicates that in coupled chaotic continuous systems, blowout bifurcation is a possible route to hyperchaos.

As in Ref. [10], the systems we consider are a class of drive-response systems. The drive system we use is a family of generalized Lorenz systems introduced by Lü and Chen[24] in 2001. It is described by

$$\begin{aligned} \dot{x} &= (25\alpha + 10)(y - x), \\ \dot{y} &= (28 - 35\alpha)x + (29\alpha - 1)y - xz, \\ \dot{z} &= xy - \frac{1}{3}(\alpha + 8)z. \end{aligned} \tag{1}$$

System (1) has the following special features: (i) system (1) is a chaotic system when $\alpha \in [0, 1]$; (ii) system (1) is very simple and has only one key parameter α; (iii) system (1) connects Lorenz's and Chen's systems; (iv) the control of the parameter α in system (1) reveals the evolution of dynamical behaviours from Lorenz attractor to Chen attractor.

For the response subsystem, we construct the two-dimensional response subsystem B as follows:

$$\dot{u} = v, \quad \dot{v} = -u^3 + pSu - 0.6v, \tag{2}$$

where S is a signal from a drive system, p is the coupling parameter. If we set $\dot{u} = \dot{v} = 0$, we obtain a fixed point of system (2) at $u = v = 0$.

*Supported in part by the National Natural Science Foundation of China (Nos. 60774088, 10772135), the Program for New Century Excellent Talents in University of China (NCET), the Science and Technology Research Key Project of Education Ministry of China (No.107024), and the Frontier Technology Research Project of Tianjin (Nos. 08JCZDJC21900, 07CYBJC05800).

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For the model, we take the variable x of system (1) as the drive signal of system (2) and obtain the following five-dimensional on–off intermittent system:

$$\begin{aligned}\dot{x} &= (25\alpha + 10)(y - x),\\ \dot{y} &= (28 - 35\alpha)x + (29\alpha - 1)y - xz,\\ \dot{z} &= xy - \frac{1}{3}(\alpha + 8)z,\\ \dot{u} &= v, \quad \dot{v} = -u^3 + pxu - 0.6v. \end{aligned} \tag{3}$$

Since system (2) is uni-directionally coupled by system (1), the dynamics of variables x, y and z are independent of those of variables u and v. Moreover, the system is symmetrical with respect to $(u, v) \to (-u, -v)$, therefore $u = v = 0$ is an invariant three-dimensional manifold in the phase space. Within this invariant subspace, there is a chaotic attractor of system (1). Here we take p as the control parameter of the system.

If we vary the coupling parameter p to make the chaotic invariant manifold $u = v = 0$ change its transverse stability, blowout bifurcation occurs.

In our numerical studies, we find that as p varies, in the parameter range of $\alpha \in [0, 1]$, the systems described by system (3) has some common characters as follows.

As p varies, the transverse stability of the invariant manifold $u = v = 0$ changes. The stability can be measured by the transverse Lyapunov exponent $\lambda_\perp$ of system (3). Take $(\delta u, \delta v)$ as an infinitesimal perturbation transverse to $u = v = 0$. According to system (2), the motion equation of the perturbation $(\delta u, \delta v)$ near the invariant subspace is

$$\dot{\delta} u = \delta v, \quad \dot{\delta} v = px\delta u - 0.6\delta v. \tag{4}$$

The exponent $\lambda_\perp$ is calculated via

$$\lambda_\perp = \lim_{t\to\infty} \frac{1}{t} \ln[\delta(t)/\delta(0)], \tag{5}$$

where $\delta(t) = \{[\delta u(t)]^2 + [\delta v(t)]^2\}^{1/2}$. If $\lambda_\perp$ is negative, the invariant manifold $u = v = 0$ is transversely stable, and if $\lambda_\perp$ is positive, the manifold is transversely unstable. The point when $\lambda_\perp \cong 0$ is a blowout bifurcation point. If $\lambda_\perp$ is slightly positive, the phenomenon of on–off intermittency appears.

For each value in $\alpha \in [0, 1]$, we observe that blowout bifurcation occurs in the systems, and on–off intermittency appears. Figure 1 shows the positive blowout bifurcation points of system (3) versus α in $\alpha \in [0, 1]$.

Another common character of system (3) is that with the increase of p, the system undergoes a symmetric transition of hyperchaos–chaos–hyperchaos. The transition happens accompanying or after the onset of on–off intermittency.

In particular, when $\alpha \in [0.25, 0.6]$, the transition points coincide with the blowout bifurcation points, as shown in Figs. 1 and 2. In Fig. 1, the blowout bifurcation points marked by asterisks are also chaos–hyperchaos transition points. When $\alpha \in [0, 0.25)$ or $\alpha \in (0.6, 1]$, after the blowout bifurcation, with the increase of p, the systems transit from chaos to hyperchaos as shown in Fig. 3.

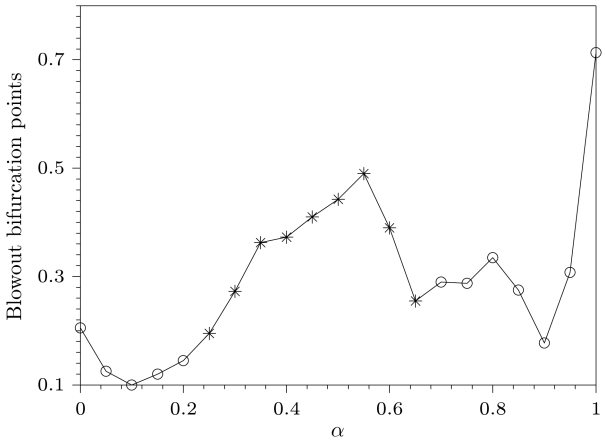

Fig. 1. Blowout bifurcation points of system (3) versus the coupling parameter α when $\alpha \in [0, 1]$.

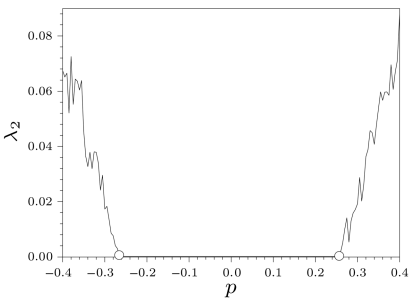

Fig. 2. Second largest Lyapunov exponent λ_2 versus the control parameter p for $p \in [-0.4, 0.4]$ in system (3) when $\alpha = 0.3$. The two circles are the blowout bifurcation points which coincide with the transition points of λ_2.

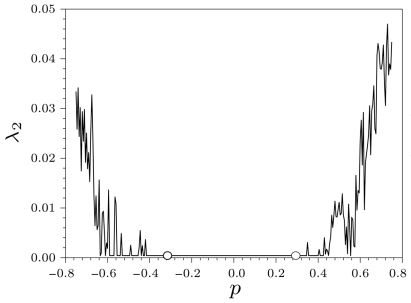

Fig. 3. Second largest Lyapunov exponent λ_2 versus p for $p \in [-0.8, 0.8]$ in system (3) when $\alpha = 0.85$. The two circles are the blowout bifurcation points. Here λ_2 becomes positive after the blowout bifurcations.

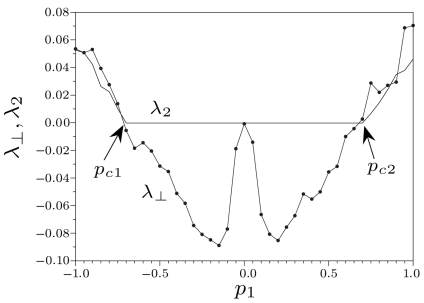

Fig. 4. The largest transverse Lyapunov exponent $\lambda_\perp$ and the second largest Lyapunov exponent λ_2 versus the control parameter p for $p \in [-1, 1]$ in system (6).

From Figs. 2 and Fig. 3, we see that the transition process is almost symmetric, i.e. if $p = p_1$ is a chaos–hyperchaos transition point of system (3), then $p = -p_1$ is also a transition point of the system. This is because the dynamics of system (3) has a p-parameter symmetry. It is shown that the time series of x and u in system (3) are always approximately symmetrical about $x = 0$ and $u = 0$ respectively. Since $\dot{v} = -u^3 + pxu - 0.6v = -u^3 + (-p)(-x)u - 0.6v = -u^3 + (-p)x(-u) - 0.6v$, for a constant value of $p = p_0$ the dynamics of the system is approximately the same as that of it at $p = -p_0$.

As an example, when $\alpha = 1$, we take Chen's system as the drive system. The system (3) becomes

$$\dot{x} = 35(y - x), \quad \dot{y} = -7x + 28y - xz, \quad \dot{z} = xy - 3z,$$
$$\dot{u} = v, \quad \dot{v} = -u^3 + pxu - 0.6v. \tag{6}$$

A plot of $\lambda_\perp$ and λ_2 of system (6) versus the coupling coefficient p for $p \in [-1, 1]$ is shown in Fig. 4. We see $\lambda_\perp$ passes through zero at $p_{c1} \approx -0.702$ and $p_{c2} \approx 0.713$. Thus there symmetrically exists two blowout bifurcation points.

While p takes values corresponding to slightly positive $\lambda_\perp$, we observe typical on–off intermittency in $u(t)$ and $v(t)$, as shown in Fig. 5.

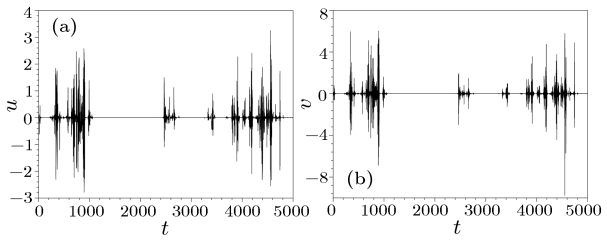

Fig. 5. (a) Time series of $u(t)$; (b) time series of $v(t)$ with $p = 0.8$ in system (6). The two time series exhibit typical on–off intermittency.

The statistical distribution of laminar phase length T of the intermittency in system (6) is investigated. We collect about 18000 laminar phases from time series of $u(t)$ for $p = 0.27$. As can be seen in Fig. 6, the distribution agrees with a $-3/2$ power-law scaling in Ref. [26], which is the statistical characteristic of on–off intermittency.

From the variation of λ_2 as shown in Fig. 4 (solid line), it is observed clearly that with the increase of p, system (6) undergoes a symmetric hyperchaos–chaos–hyperchaos transition. We see that λ_2 passes through zero at critical values very close to p_{c1} and p_{c2}. Thus the two blowout bifurcation points coincide with the two transition points of chaos–hyperchaos. When $p < p_{c1}$ or $p > p_{c2}$, the attractor is a hyperchaotic one. When $p_{c1} < p < p_{c2}$, the attractor is chaotic.

Figure 7 shows the time series of u and v before the blowout bifurcation. We can see that the subsystem $B : \{u, v\}$ stabilizes asymptotically at its fixed point $(u = 0, v = 0)$. After the blowout bifurcation the fixed point loses its stability, the time series of u and v exhibit on–off intermittent bursting (Fig. 5), and the chaotic attractor within the invariant subspace becomes a bursting hyperchaotic attractor as shown in Fig. 8(b).

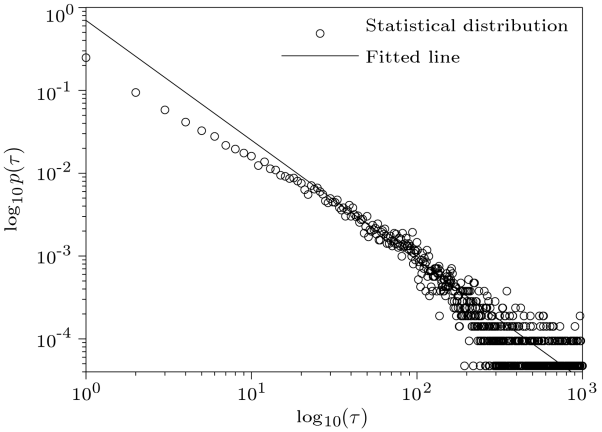

Fig. 6. Distribution of the laminar duration time of the intermittency in system (6). The straight line is drawn from the least-squares method and has a slope of -1.463.

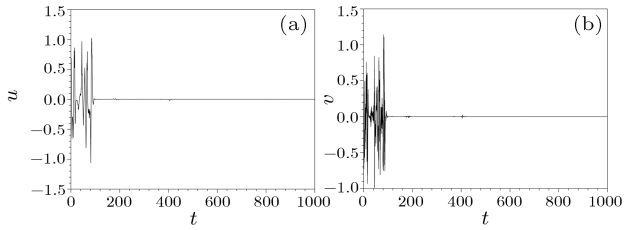

Fig. 7. (a) Time series of $u(t)$; (b) time series of $v(t)$ for $p = 0.45$ in system (6). They stabilize at $u(t) = 0$ and $v(t) = 0$.

Next, we make an analysis of the hyperchaos–chaos–hyperchaos transition process.

Like the example of system (6), for all $\alpha \in [0, 1]$ before the bifurcation, the transverse system $B : \{u, v\}$ of system (3) stabilizes asymptotically at its fixed point $(u = 0, v = 0)$. After the blowout bifurcation the fixed point loses its stability, the time series of u and v exhibit on–off intermittent bursting.

From the Lyapunov exponent spectrum, we observe that among the five Lyapunov exponents of system (3), the largest one λ_1 and the smallest one λ_5 almost remain unchanged, while the other three exponents change continuously as p varies. These two exponents are exactly corresponding to those of the drive subsystem A. That is to say, the change of the coupling parameter p only has impact on the stability of the transverse system of system (3). When $\lambda_\perp$ changes from negative to positive, the subsystem $B : \{u, v\}$ loses its stability and the invariant manifold $u = v = 0$ of system (3) loses its transverse stability.

After the bifurcation in system (3), further de-

crease (or increase) of p will cause the transverse system more and more unstable. To a certain extent, the subsystem $B : \{u, v\}$ becomes chaotic, leading to an additional positive Lyapunov exponent for the whole system. Therefore, the whole system transits to hyperchaos from chaos. When $\alpha \in [0.25, 0.6]$, the transition happens almost the same time with the blowout bifurcation; when $\alpha \in [0, 0.25)$ or $\alpha \in (0.6, 1]$, after the blowout bifurcation and the instability of the invariant manifold $u = v = 0$ increasing to some extent, the chaos–hyperchaos transition happens.

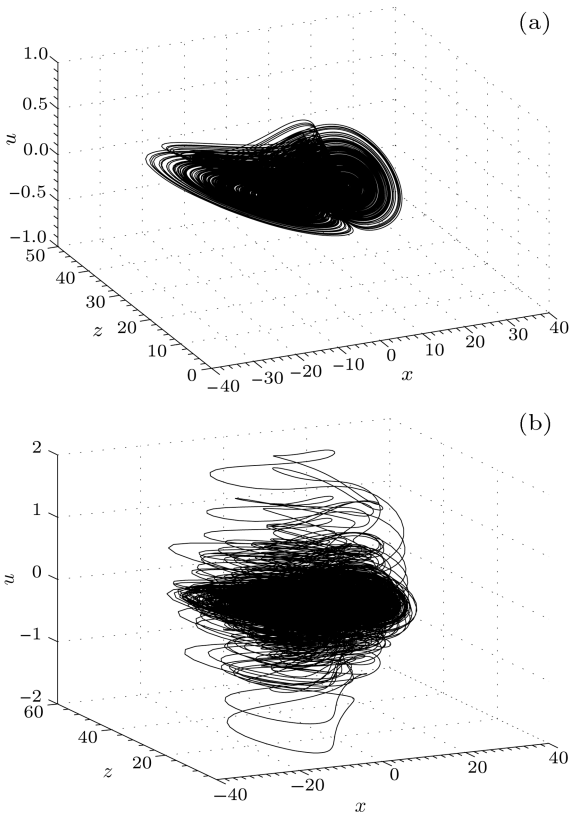

Fig. 8. Three-dimensional projections of the attractors of system (6), (a) chaotic attractor before the blowout bifurcation at $p = 0.4$; (b) hyperchaotic attractor after the bifurcation at $p = 0.8$

The transition confirms the general route to high-dimensional chaos presented by Harrison and Lai.[27] A characteristic feature of this route is that the second largest Lyapunov exponent passes through zero continuously as shown in Fig. 5. This gives evidence that the transition is continuous and smooth.

According to the theory of unstable periodic orbits, chaos–hyperchaos transition is caused by changes in the stability of an infinite number of unstable periodic orbits embedded in the chaotic attractor.[23] Blowout bifurcation is mediated by changes in the transverse stability of an infinite number of unstable periodic orbits embedded in the chaotic attractor.[22] It is likely that in a dynamical system, the loss of local stability results in the change of the stability of the whole system, like the route towards chaos through local bifurcation. We conclude that the change in the transverse stability of unstable periodic orbits embedded in the chaotic attractor induces the loss of the global stability of the unstable periodic orbits like the chaos–hyperchaos transition in system (3). Thus in a coupled chaotic system, the occurrence of blowout bifurcation may indicate a chaos–hyperchaos transition of the whole system. Our work indicates that in coupled chaotic continuous systems, blowout bifurcation is a possible route to hyperchaos.

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