Exact Solutions of (2+1)-Dimensional Euler Equation Found by Weak Darboux Transformation *

LOU Sen-Yue(楼森岳)[1,2**], LI Yi-Shen(李翊神)[1,3]

[1] *Center of Nonlinear Science, Ningbo University, Ningbo 315211*

[2] *Department of Physics, Shanghai Jiao Tong University, Shanghai 200030*

[3] *Department of Mathematics, and Center of Nonlinear Science, University of Science and Technology of China, Hefei 230026*

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The weak Darboux transformation of the (2+1) dimensional Euler equation is used to find its exact solutions. Starting from a constant velocity field solution, a set of quite general periodic wave solutions such as the Rossby waves can be simply obtained from the weak Darboux transformation with zero spectral parameters. The constant vorticity seed solution is utilized to generate Bessel waves.

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To find exact solutions of the (2+1)-dimensional Euler equation is an interesting and important issue in mathematics, fluid mechanics,[1] plasma physics,[2,3] oceanography,[4] atmospheric dynamics,[5,6] superfluid and superconductivity,[7] cosmography and astrophysics,[8] statistical physics,[9] field theory, particle physics[10] and condense matters.[11−13]

Because of the importance of the issue, there are a large number of papers on the solutions of the Euler equation and the related Navier–Stokes equation. To reach some *special* exact analytical solutions, one usually use the standard group theory.[14] However, in most cases, one has to study the problems numerically because of the difficulty to find more exact solutions of the nonlinear Euler equation.[15] In 2001,[16] Li found a (weak) Lax pair for the Euler equation and a special (weak) Darboux transformation (DT) of the Lax pair is obtained. Usually, a DT can be recursively find new solutions from a known one. However, one fails to reach new exact solutions from the DT of Ref. [16]. In this Letter, we try to find some new exact solutions starting from the DT in its new form.

In Ref. [16], the author wrote the (2+1)-dimensional Euler equation in its vorticity form

$$\Omega = \psi_{xx} + \psi_{yy} \equiv \Delta\psi, \tag{1}$$

$$\Omega_t + [\psi, \Omega] = 0, \ [\psi, \Omega] \equiv \psi_x\Omega_y - \psi_y\Omega_x, \tag{2}$$

where Ω is the vorticity, ψ is the stream function, and the velocity field $\{u, v\}$ is linked to the vorticity and the stream functions by

$$u = -\psi_y, \ v = \psi_x, \ \Omega = v_x - u_y = \Delta\psi. \tag{3}$$

The weak Lax pair for Eq. (2) has been given as follows:

$$L\varphi = \lambda\varphi, \ L\varphi \equiv [\Omega, \varphi], \tag{4}$$

$$\varphi_t + A\varphi = 0, \ A\varphi \equiv [\psi, \varphi]. \tag{5}$$

We should point out that this Lax pair is weak since it can only be used to prove[17]

$$[\Omega_t + [\psi, \Omega], \varphi] = 0. \tag{6}$$

It has to be also emphasized that Eq. (1) is a definition equation of Ω with respect to ψ and can not be derived from the Lax pair (4) and (5).

In Ref. [17], the authors have also given a (weak) DT for Eq. (2) with the zero spectral one. However, up to date, there has been no one to construct new solutions by means of the weak Darboux transformation of Ref. [17].

Recently, we have obtained a new weak DT of Eq. (2) with three spectral parameters. When the spectral parameters are all taken to be zero, the new DT is reduced to an equivalent form of Ref. [17]. However the new form is much easier to be applied to find exact solutions than the original one. In this study, we apply this transformation to generate new exact solutions of the (2+1)-dimensional Euler equation. For self-containing, we first prove the proposition of the new weak DT and then choose different seeds to produce some new exact explicit solutions.

Proposition 1. If $\{\Omega, \psi, \varphi\}$ is a solution of the Euler equations (1) and (2) and the Lax pair (4) and (5), and f is an arbitrary solution of the Lax pair with the spectral parameter λ_0, $G = G(f)$ being an arbitrary function of f, then $\{\Omega_1, \psi_1, \phi_1\}$ with the definition

$$\psi_1 = \psi + p, \quad \Omega_1 = \Omega + q, \quad \phi_1 = G(f)\varphi \tag{7}$$

is a solution of Eq. (6) with $\psi \to \psi_1, \varphi \to \phi_1$ and the Lax pair

$$[\Omega_1, \phi_1] = \lambda_1\phi_1, \quad \phi_{1t} + [\psi_1, \phi_1] = 0, \tag{8}$$

if p and q are determined by

$$[q, G\varphi] = (\lambda_1 - \lambda)G\varphi - G_f\lambda_0 f\varphi, \ [p, G\varphi] = 0, \tag{9}$$

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** To whom correspondence should be addressed. Email: sylou@mail.sjtu.edu.cn

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$$q = \Delta p. \tag{10}$$

Proof: To prove the proposition 1, we only need to prove that $\{\Omega_1, \psi_1, \phi_1\}$ satisfy the Lax pair (8). From the first one of Eq. (8), we have

$$\begin{aligned}[\Omega_1, \phi_1] &= [\Omega + q, G\varphi] = [\Omega, G\varphi] + [q, G\varphi] \\ &= G[\Omega, \varphi] + G_f[\Omega, f]\varphi + [q, G\varphi] = \lambda_1 G\varphi.\end{aligned}$$

By using

$$[\Omega, f] = \lambda_0 f, \quad [\Omega, \varphi] = \lambda\varphi,$$

it yields the first formula of Eq. (9). In the same way, from the second one of Eq. (8), we have

$$\begin{aligned}\phi_{1t} + [\psi_1, \phi_1] &= G_f f_t \varphi + G\varphi_t + [\psi, G\varphi] + [p, G\varphi] \\ &= G_f f_t \varphi + G\varphi_t + G_f[\psi, f]\varphi \\ &\quad + G[\psi, \varphi] + [p, G\varphi] = 0.\end{aligned}$$

It is obvious that the use of

$$f_t + [\psi, f] = 0, \quad \varphi_t + [\psi, \varphi] = 0,$$

arrives at the second formula of Eq. (1). The relation (10) just comes from the definition equation (1). While Eq. (6) with $\psi \to \psi_1, \varphi \to \phi_1$ is only a consistent condition of Eq. (8). The proposition 1 is then proven.

Because the Lax pair is weak, the DT given here is only under the weak meaning.

Though the DT is only a weak one, it can still be used to find exact solutions of the Euler equation. Here are some interesting examples.

Example 1. Rossby waves and its generalization from the constant flow. It is clear that for the (2+1)-dimensional Euler equations (1) and (2), there is a constant flow solution

$$u = -b, \ v = a, \tag{11}$$

i.e.

$$\psi = ax + by, \quad \Omega = 0, \tag{12}$$

where a and b are arbitrary constants.

Substituting the seed solution (11) into the space part of the Lax pair,

$$L\varphi = \lambda\varphi, \ Lf = \lambda_0 f, \tag{13}$$

leads to

$$\lambda = \lambda_0 = 0.$$

From

$$\varphi_t + A\varphi = 0, \quad f_t + Af = 0 \tag{14}$$

and the seed solution (11), one can obtain that φ and f are arbitrary functions of $\xi = ax + by$ and $\eta = bt + x$, i.e.

$$\varphi = F_1(\xi, \eta), \quad f = F_2(\xi, \eta), \tag{15}$$

where F_1 and F_2 are arbitrary functions of $\{\xi, \eta\}$.

When $\lambda_1 = 0$, from the transformation theorem it is easy to obtain

$$q = Q(F(\xi, \eta)), \quad p = P(F(\xi, \eta)) \tag{16}$$

from Eq. (9), where P and Q are two arbitrary functions of $F = F(\xi, \eta)$ and

$$F \equiv G(f)\varphi \equiv G(F_2)F_1.$$

Because the DT theorem is only a weak one, in order to ensure that the result is a solution of the Euler equation, we should substitute the result to the original Euler equation to verify whether the result is really a solution of the Euler equation. For simplicity of calculation, we rewrite the Euler equation system (1) and (2) in the coordinates $\{\xi, \eta, t\}$ for the new solution $\{\Omega_1, \psi_1\}$:

$$E_1 \equiv \Omega_1 = (a^2 + b^2)\psi_{1\xi\xi} + \psi_{1\eta\eta} + 2a\psi_{1\xi\eta} = 0, \tag{17}$$

$$E_2 \equiv \Omega_{1t} + b\Omega_{1\xi}\psi_{1\eta} - b\Omega_{1\eta}\psi_{1\xi} + b\Omega_\eta = 0. \tag{18}$$

It is straightforward to verify that E_2 is satisfied identically for Eq. (7) with the seed Eqs. (12) and (16), which is also guaranteed by Proposition 1.

Without loss of generality, one of the arbitrary functions P and Q given in Eq. (16) can be fixed to a convenient form

$$P = F(\xi, \eta). \tag{19}$$

Under the simplification (16), the final result can be summarized as the following conclusion.

If

$$\begin{aligned}H(X, Y) &\equiv H\Big(\frac{1}{b}[\xi - (a+b)\eta], \frac{1}{b}[\xi - (a-b)\eta]\Big) \qquad (20)\\ &\equiv H\big(x + y - (a-b)t, y - x - (a+b)t\big) \\ &\equiv F(\xi, \eta) \qquad (21)\end{aligned}$$

is a solution of

$$H_{XX} + H_{YY} = Q(H), \tag{22}$$

then

$$\Omega_1 = Q(H), \quad \psi_1 = ax + by + H, \tag{23}$$

is a solution of the Euler equation system (1) and (2), where Q is an arbitrary function of H. The simplest periodic wave solution, the Rossby wave solution,

$$\Omega_1 = \frac{B(k_1^2 + k_2^2)}{2b^2}\cos(k_1 x + k_2 y + (k_1 b - k_2 a)t + x_0), \tag{24}$$

$$\psi_1 = ax + by + B\cos(k_1 x + k_2 y + (k_1 b - k_2 a)t + x_0), \tag{25}$$

is related to the special selection

$$Q(F_1) = \frac{(k_1^2 + k_2^2)}{2b^2}F_1. \tag{26}$$

In Fig. 1, the structure of the stream function of the Rossby wave (25) is plotted with particular parameter selections,

$$B = 1, \quad x_0 = 0, \quad a = b = k_1 = k_2 = 0.01. \tag{27}$$

Example 2. Bessel waves from constant vorticity. It is also easy to verify that the (2+1)-dimensional

Euler equations (1) and (2) possess a linearly sheared flow solution

$$u = -2y, \quad v = 2x, \tag{28}$$

i.e. the constant vorticity solution,

$$\psi = x^2 + y^2, \quad \Omega = 4, \tag{29}$$

where the vorticity possesses a special constant value 4.

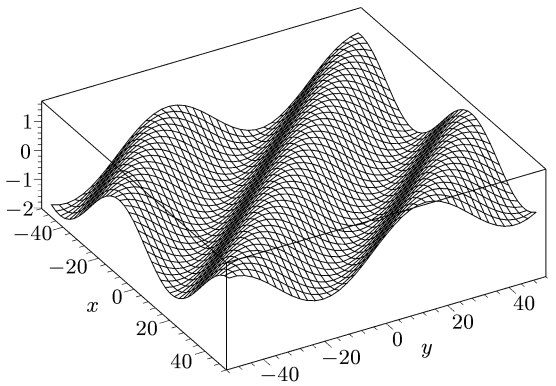


Fig. 1. A special plot of the periodic wave for the stream function ψ given by Eq. (25) with the special parameter selections (27) at $t = 0$.

Substituting the seed solution ((28) into the space part of the Lax pair (13) leads to

$$\lambda = \lambda_0 = 0.$$

From the time part of the Lax pair (14) and the seed solution (28), we find that φ and f are arbitrary functions of

$$r = \sqrt{x^2 + y^2}, \tag{30}$$

$$\theta = t + \frac{1}{2}\arctan\frac{x}{y}, \tag{31}$$

namely,

$$\varphi = F_1(r, \theta), \quad f = F_2(r, \theta), \tag{32}$$

where F_1 and F_2 are arbitrary functions of $\{r, \theta\}$.

If $\lambda_1 = 0$, it is easy to obtain

$$q = Q(F(r, \theta)), \quad p = P(F(r, \theta)) \tag{33}$$

from Eq. (9), where P and Q are the two new arbitrary functions of $F \equiv F(r, \theta) \equiv G(f)\varphi = G(F_2(r, \theta))F_1(r, \theta)$.

Similar to Example 1, we rewrite the Euler equation system (1) and (2) in the new coordinate system $\{r, \theta\}$ for the new solution $\{\Omega_1, \psi_1\}$:

$$\Gamma_1 \equiv \Omega_1 - \psi_{1rr} - \frac{1}{r}\psi_{1r} - \frac{1}{r^2}\psi_{1\theta\theta} = 0, \tag{34}$$

$$\Gamma_2 \equiv \Omega_{1t} + \Omega_{1\theta} + \frac{1}{2r}(\Omega_{1r}\psi_{1\theta} - \Omega_{1\theta}\psi_{1r}) = 0. \tag{35}$$

It is evident that Γ_2 is identically satisfied for Eq. (7) with the seed (29) and (33), which is also guaranteed by Proposition 1.

Similarly, without loss of generality, we can take

$$P(F) = F(r, \theta) \tag{36}$$

for simplicity. Under the simplification (36), the final result can be summarized as follows.

If $F \equiv F(r, \theta)$ is a solution of

$$F_{rr} + \frac{1}{r}F_r + \frac{1}{4r}F_{\theta\theta} = Q(F), \tag{37}$$

then

$$\Omega_1 = 4 + Q(F), \quad \psi_1 = r^2 + F, \tag{38}$$

is a solution of the Euler equation system (1) and (2), where Q is an arbitrary function of F.

It is worthwhile to write down two special cases of this example.

Case 1. General constant vorticity, $\Omega_1 = B, Q(F) = B - 4$. In this case, the general solution has the form

$$\Omega_1 = B, \quad i = \sqrt{-1} \tag{39}$$

$$\psi_1 = \frac{B}{4}(x^2 + y^2) + g_1(z) + g_2(z^*), \tag{40}$$

where

$$z \equiv 4t + 2\arctan\frac{x}{y} - i\ln(x^2 + y^2), \tag{41}$$

z^* is the complex conjugate of z; g_1 and g_2 are arbitrary complex functions of z and z^*, respectively. Obviously, when we take g_2 as the complex conjugate function of g_1, then the solution (40) is real.

Case 2. $Q(F) = -A^2F$, with constant A. In this special case, Eq. (37) can be solved via variable separation approach and the final result reads

$$\Omega_1 = 4 - A^2\sum_{i=1}^{N}\cos(2c_i(\theta - \theta_i)) \cdot [C_{1i}J_{c_i}(Ar) + C_{2i}N_{c_i}(Ar)], \tag{42}$$

$$\psi_1 = r^2 + \sum_{i=1}^{N}\cos(2c_i(\theta - \theta_i)) \cdot [C_{1i}J_{c_i}(Ar) + C_{2i}N_{c_i}(Ar)], \tag{43}$$

where N is an arbitrary integer; c_i, C_{1i}, C_{2i} and θ_i all are arbitrary constants; $J_{c_i}(x)$ and $N_{c_i}(x)$ are c_ith Bessel and Neumann functions of x; r and θ are given by Eqs. (30) and (31), respectively.

In Fig. 2, four special Bessel waves are plotted for the stream function ψ

$$\psi = (x^2 + y^2) + 100\cos\left[2c\left(t + \frac{1}{2}\arctan\frac{x}{y}\right)\right] \cdot J_c(2\sqrt{x^2 + y^2}) \tag{44}$$

for $c = 0, 1, 3$ and 8, respectively, at time $t = 0$.

Though the (weak) Lax pair and the first weak DT theorem with zero spectral parameter of the (2+1)-dimensional Euler equation have been given by Li *et al.* for five years,[16,17] there is no one to use them to obtain exact solutions of the model.

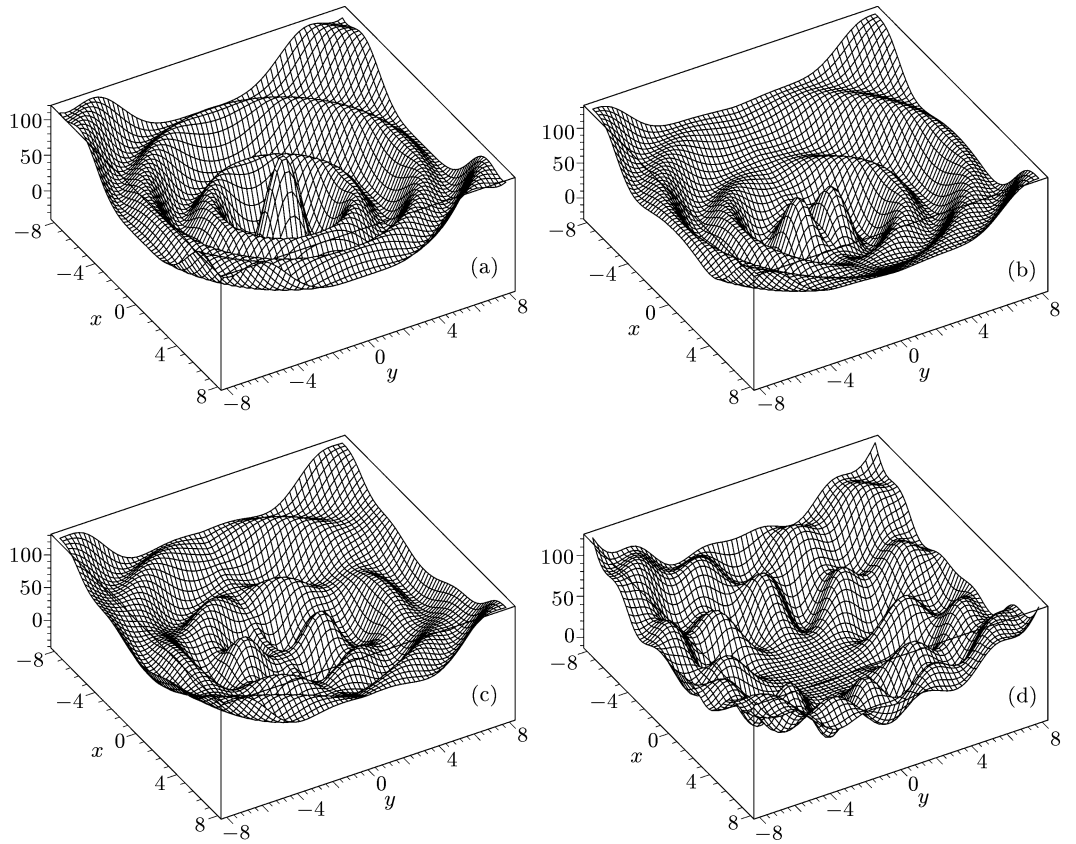


Fig. 2. Four particular Bessel waves for the stream function ψ given by Eq. (44) at $t = 0$ with (a) $c = 0$, (b) $c = 1$, (c) $c = 3$ and (d) $c = 8$.

A brief report on the new forms of the weak Darboux transformations has been given in Ref. [18] without proofs and examples for the both (2+1)- and (3+1)-dimensional Euler equations. In this study, we make use of the new form of the weak DT theorem to construct exact solutions of the (2+1)-dimensional Euler equation.

Starting from the constant flow solution, the weak DT theorem generates a quite general solution taking the Rossby waves as its special case.

Combining the constant vorticity seed solution and the weak DT theorem results in another kind of waves embodying various Bessel waves.

Because of the wide applications of the Euler equations, more exact solutions yielded via the weak Darboux transformations for both (2+1)- and the (3+1)-dimensional models are worth studying further.

References

[1] Chavanis P H and Sommeria J 1997 *Phys. Rev. Lett.* **78** 3302
Chavanis P H 2000 *Phys. Rev. Lett.* **84** 5512
[2] Grasso D et al 2001 *Phys. Rev. Lett.* **86** 5051
[3] Bergmans J and Schep T J 2001 *Phys. Rev. Lett.* **87** 195002
[4] Marshall J et al 1997 *J. Geophys. Res. Oceans* (C3) **102** 5753
Canuto V M and Dubovikov M S 2005 *Ocean Modelling* **8** 1
[5] Girard C, Benoit R, Desgagne M 2005 *Monthly Weather Rev.* **133** 1463
[6] Huang T S, Ho C W and Alexander C J 1998 *J. Geophys. Res.: Planets* (E9) **103** 20267
[7] Haldane F D M and Wu Y 1985 *Phys. Rev. Lett.* **55** 2887
[8] Xu C M et al 2005 *Phys. Rev.* D **71** 024030
[9] Niemi A J 2005 *Phys. Rev. Lett.* **94** 124502
[10] Faddeev L, Niemi A J and Wiedner U 2003 hep-ph/0308240
[11] Leggett A J 2001 *Rev. Mod. Phys.* **73** 307
[12] Chuang I et al 1991 *Science* **251** 1336
[13] Babaev E, SudbøA and Ashcroft N W *Nature* **431** 666
[14] Kapitanskiy L V 1978 *Dokl. Acad. Sci. USSR* **243** 901
Fushchych W I et al 1991 *J. Phys.* A **24** 971
[15] Alekseev A K and Navon I M 2005 *Int. J. Num. Meth. Fluids* **47** 45
Zhang P W 1996 *J. Comput. Math.* **14** 213
Wang Y P 2005 *Appl. Math. Mech.* **26** 865
[16] Li Y G 2001 *J. Math. Phys.* **42** 3552
[17] Li Y G and Yurov A V 2003 *Stud. Appl. Math.* **111** 101
Li Y C and Shvidkoy R 2004 *J. Math. Anal. Appl.* **292** 311
[18] Lou S Y et al 2005 nlin.PS/0509039