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EXACT FORMULAS OF NON-LINEAR STANDING WAVES

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On the basis of hydrodynamic equations, an exact theory of non-linear standing sound waves is proposed, avoiding the unaccountable results of earlier theories.

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Intense sound waves not only cause acoustic fatique to airframes and missiles, malfunctions to sensitive instruments, but also serve as powerful tools in sonic and ultrasonic mechanical and chemical treatments, in medical treatments, and in the study of solids. The finite-amplitude sound waves are interesting theoretically as well as in practice. The study aroused the attention of scientists from early times[1] and advanced vigorously in recent years, opening up new frontiers in physics[2]. The problem of propagation of finite-amplitude progressive waves has been treated with great success, and theory thereof established. But in case of standing waves, further studies are badly needed and the result of everincreasing distortion, as a consequence of treatment with perturbation technique[3], is often questioned from physical ground. It is the purpose of the present work to reconsider this problem and try to find a rigorous solution thereof.

Limit our attention to lossless medium, air, to make the issue clear. Start from the Euler formulation of the equations of motion and continuity,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \quad , \tag{1}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \quad , \tag{2}$$

where u, p, ρ are, respectively, the particle velocity, sound pressure and the density of air. Introducing the new thermodynamic quantity

$$\lambda = \int_{\rho_0}^{\rho} \frac{c}{\rho} d\rho = \int_0^p \frac{1}{\rho c}\, dp \quad , \tag{3}$$

c being the velocity of sound, the eqs.(1) and (2) are transformed into

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + c\frac{\partial \lambda}{\partial x} = 0 \quad , \tag{4}$$

$$\frac{\partial \lambda}{\partial t} + u\frac{\partial \lambda}{\partial x} + c\frac{\partial u}{\partial x} = 0 \quad . \tag{5}$$

Taking the sum and difference of these two equations, it yields

$$\frac{\partial r}{\partial t} + (u + c)\,\frac{\partial r}{\partial x} = 0 \quad , \tag{6}$$

$$\frac{\partial s}{\partial t} + (u - c)\,\frac{\partial s}{\partial x} = 0 \quad , \tag{7}$$

where $2r = \lambda + u$ and $2s = \lambda - u$ are the Riemann invariants. It is seen that r and s represent wave motions propagating in the positive and negative x-directions, respectively. The corresponding "local" or phase velocities are $c \pm u$. It can be shown that the velocity of sound c, at finite amplitudes, depends on the direction of propagation, and is $c_0 \pm (\gamma - 1)u/2$ for the positive and negative waves, respectively. Substituting, the eqs.(6) and (7) become

$$\frac{\partial r}{\partial t} + \left(\frac{\gamma + 1}{2}u + c_0\right)\,\frac{\partial r}{\partial x} = 0 \quad , \tag{8}$$

$$\frac{\partial s}{\partial t} + \left(\frac{\gamma + 1}{2}u - c_0\right)\,\frac{\partial s}{\partial x} = 0 \quad . \tag{9}$$

At this point, the special character of the standing waves must be stressed. Firstly, the principle of superposition for simple systems is not valid for nonlinear acoustics, because the r and s waves affect each other through the vibrating velocity u, which is the particle velocity of the composite standing wave, not that of r or s wave. And further, no where on the waveform is u a constant, as in travelling wave, it varies with time and space, and does not propagate. The phase velocities, $c_0 \pm (\gamma + 1)u/2$, must be the instantaneous velocity. The phase change due to c_0 is $c_0 t$ as usual, but that due to u should be $\int u\, dt = \xi$, the particle displacement on the standing wave. For sinusoidal excitation, the Riemann simple waves are then

$$r = \frac{1}{2}u_0 \cos\left(\omega t - kx + \frac{\gamma + 1}{2}\,k\xi\right) \tag{10}$$

and

$$s = \frac{1}{2}u_0 \cos\left(\omega t + kx - \frac{\gamma + 1}{2}\,k\xi\right) \quad , \tag{11}$$

where $\omega = kc_0$ is the angular frequency and k the wave number. And the velocity wave is, then,

$$u = r - s = u_0 \sin\omega t \sin k\left(x - \frac{\gamma + 1}{2}\,\xi\right) \quad . \tag{12}$$

Substituting into eq.(1) and solving for p, it yields the sound pressure

$$p = p_0 \cos\omega t \cos k\left(x - \frac{\gamma+1}{2}\xi\right) \quad , \tag{13}$$

where $p_0 = u_0\rho_0 c_0$ is the amplitude of the pressure wave. Equations (12) and (13) are both exact. But they both depend on the displacement ξ, which may be derived from the relation $u = \partial\xi/\partial t$, and it yields

$$\begin{aligned}\xi = A\cos\omega t \sin kx - \frac{\gamma+1}{4} kA^2 \cos^2\omega t \cos kx \sin kx \\ + \frac{(\gamma+1)^2}{24} k^2A^3 \cos^3\omega t \cos 2kx \sin kx + \cdots\end{aligned} \tag{14}$$

where $A = -u_0/\omega$, or $kA = -u_0/c_0 = -p_0/\gamma P_0$, P_0 being the atmospheric pressure. Explicit forms of u and p may be found by substituting this equation of displacement into eqs.(12) and (13) expanding into power series as

$$\begin{aligned}u = u_0 \sin\omega t \sin kx + \frac{\gamma+1}{2}\frac{u_0^2}{c_0} \cos\omega t \sin\omega t \cos kx \sin kx \\ + \frac{(\gamma+1)^2}{8}\frac{u_0^3}{c_0^2} \cos^2\omega t \sin\omega t \cos 2kx \sin kx + \cdots\end{aligned} \tag{15}$$

and

$$\begin{aligned}p = p_0 \cos\omega t \cos kx + \frac{\gamma+1}{2}\frac{p_0^2}{\gamma P_0} \cos^2\omega t \sin^2 kx \\ - \frac{(\gamma+1)^2}{4}\frac{p_0^3}{(\gamma P_0)^2} \cos^3\omega t \sin^2 kx \cos 2kx + \cdots\end{aligned} \tag{16}$$

For the standing wave between two rigid walls at a distance L apart, the boundary conditions are $u = 0$ at $x = 0$ and L. This results

$$k = n\pi/L \ , \quad n = \text{integer} \quad . \tag{17}$$

And for a closed tube driven with a piston, the boundary conditions are, then, $u = 0$ at $x = 0$ and $u = U\sin\omega t$ at $x = L$, and one has

$$u_0 = \frac{U}{\sin kL} \quad . \tag{18}$$

Substituting these values into eqs.(14)–(16) results the standing waves for these two related cases.

In a closed tube, the amplitudes, u_0, p_0, and A may rise to rather large values near resonance, only limited by the absorption at the end of the tube. It can be shown that, when the frequency is close to resonance, the denominator, instead of $\sin kL$, should take the exact value

$$\left[\sin^2(kL+b)+g^2\right]^{1/2} \quad , \tag{19}$$

where $g - jb = \rho_0 c_0 / Z_A$, the relative acoustical admittance of the material forming the end. The resonance happens at $kL + b = n\pi$, and the denominator takes the value g in magnitude, which is usually very small for solid material.

It may be concluded from eqs.(15) and (16) that the distortion of finite-amplitude standing waves, being of the order u_0/c_0 or $p_0/\gamma P_0$ compared to the fundamental, is usually rather small, as the ratio is only one tenth at a level 174dB of sound pressure. Experimental results of earlier authors[3,4] showed little distortion before shock waves set in, agreeing with the present conclusion.

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