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A SIMPLE DERIVATION OF FIELD THEORY OF ISING QUANTUM CHAIN

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A simple derivation of the field theory of Ising quantum chain shows that the Ising model is equivalent to the free Majorana real spinor field. At the critical point, the field equation is conformally covariant.

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The two dimensional Ising model has long been studied. This model is equivalent to the theory of free fermions. Ferrel[1] verified that the two dimensional Ising model is equivalent to the free fermion one dimensional Dirac equation. This statement, however, needs to be further improved; the more reasonable statement should read as that the two dimensional Ising model is related to the two dimensional free fermion spinor equation. In the vicinity of critical point, this equation is conformally covariant, the critical point condition then can be obtained.

The quantum chain of two dimensional Ising model has the Hamiltonian as[2]

$$H = g\sum_{1}^{N}\sigma_n^z - \sum_{1}^{N-1}\sigma_n^x\,\sigma_{n+1}^x \quad , \tag{1}$$

where σ_n^x and σ_n^z are Pauli operators, and g is the coupling constant. Let us introduce Jordan-Wigner transformation[3]

$$c_n = e^{i\pi\sum_{1}^{n-1}\sigma_j^+\sigma_j^-}\sigma_n^- \ , \quad c_n^+ = \sigma_n^+ e^{-i\pi\sum_{1}^{n-1}\sigma_j^+\sigma_j^-} \ , \tag{2a}$$

the inverse transformation is

$$\sigma_n^- = e^{-i\pi\sum_{1}^{n-1}c_j^+c_j}c_n \ , \quad \sigma_n^+ = c_n^+ e^{i\pi\sum_{1}^{n-1}c_j^+c_j} \ , \tag{2b}$$

where $\sigma_n^{\pm}$ are the creative and destructive operators respectively. From eq.(2) and the properties of $\sigma_n^{\pm}$, we have

$$\begin{aligned} \{c_m,\ c_n^+\} &= \delta_{mn} \quad , \\ \{c_m,\ c_n\} &= \{c_m^+,\ c_n^+\} = 0 \quad , \end{aligned} \tag{3}$$

where c_n and c_n^+ are Fermi operators.

Substituting eq.(2) into eq.(1), eq.(1) can be written as

$$H = 2g \sum_{1}^{N} \left(c_n^+ \, c_n - \frac{1}{2} \right) - \sum_{1}^{N-1} \left(c_n^+ \, c_{n+1} + c_{n+1}^+ \, c_n + c_n^+ \, c_{n+1}^+ + c_{n+1} \, c_n \right) \quad . \tag{4}$$

The field equation of the Ising quantum chain of eq.(4) has the form

$$\frac{dc_n}{dx} = \frac{1}{\xi} \left[H, \, c_n \right] \quad , \quad \frac{dc_n^+}{dx} = \frac{1}{\xi} \left[H, \, c_n^+ \right] \quad , \tag{5}$$

where x refers to the Euclidean "time", and ξ is an unknown constant since the Hamiltonian may in principle be multiplied by an overall factor. This overall factor can be fixed by requiring that the field eq.(5) defined by the Hamiltonian is conformally covariant[4]. From eqs.(4) and (5), and with the aid of eq.(3), we have

$$\begin{aligned} \frac{dc_n}{dx} &= -\frac{2}{\xi} (g-1) c_n + \frac{2}{\xi} \frac{dc_n^+}{dy} \\ \frac{dc_n^+}{dx} &= \frac{2}{\xi} (g-1) c_n^+ - \frac{2}{\xi} \frac{dc_n}{dy} \end{aligned} \quad . \tag{6}$$

In obtaining of eq.(6), we have used the continuous limit condition with

$$c_{n+1} + c_{n-1} = 2c_n \, , \quad c_{n+1} - c_{n-1} = 2 \frac{dc_n}{dy} \quad . \tag{7}$$

From eq.(6), it is easy to see that

$$\left[\frac{\partial^2}{\partial x^2} + \frac{4}{\xi^2} \frac{\partial^2}{\partial y^2} - \frac{4}{\xi^2} (g-1)^2 \right] c_n = 0 \quad . \tag{8}$$

The equations similar to eqs.(7) and (8) hold for c_n^+.

If we take

$$\xi = \pm 2 \quad , \tag{9}$$

$$g = 1 \quad , \tag{10}$$

eq.(8) then reduces to the two dimensional conformally covariant Klein-Gordon equation. Equation (9) fixes the value of ξ, and eq.(10) gives the critical point condition which is related to the vanishing mass term of the field equation. Without the loss of generality by taking $\xi = 2$, the conformally covariant form of eq.(6) now reads as

$$\frac{dc_n}{dx} = \frac{dc_n^+}{dy} \, , \quad \frac{dc_n^+}{dx} = -\frac{dc_n}{dy} \quad . \tag{11}$$

Equation (11) can be expressed in the real spinor form, if we introduce the real fermion operators A_n and B_n as

$$c_n = \frac{1}{\sqrt{2}} \left(A_n - iB_n \right) \quad , \quad c_n^+ = \frac{1}{\sqrt{2}} \left(A_n + iB_n \right) \quad , \tag{12}$$

where A_n and B_n satisfy the following anticommutation rules

$$\begin{aligned} &\{A_m,\ A_n\} = \{B_m,\ B_n\} = \delta_{mn} \quad , \\ &\{A_m,\ B_n\} = 0 \quad . \end{aligned} \tag{13}$$

In terms of A_n and B_n, eq.(11) can be written as

$$\left(\frac{d}{dx} - i\frac{d}{dy}\right)(A_n + B_n) = 0 \quad , \quad \left(\frac{d}{dx} + i\frac{d}{dy}\right)(A_n - B_n) = 0 \quad . \tag{14}$$

In the continuous limit, eq.(14) is identical to the conformally covariant real fermion field equation

$$\gamma^\mu \partial_\mu \psi = 0 \quad , \tag{15}$$

where $\gamma^x = \sigma^x$, $\gamma^y = \sigma^y$, and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim \begin{pmatrix} A - B \\ A + B \end{pmatrix}$.

According to eq.(15), the fields $\psi_1(z)$ and $\psi_2(\bar{z})$ are analytic functions of variable $z = x + iy$ and $\bar{z} = x - iy$ respectively, and they can be treated separately in conformal field theory. The conformal transformation in two dimensions is equivalent to the analytical transformation which is related to the Virasoro algebra[5,6]. The representations are characterised by the central charge c of the algebra. In the case of real fermion theory of eq.(15), the stress energy tensor is given by[7]

$$T(z) = -\frac{1}{2} : \psi_1(z) \frac{\partial}{\partial z} \psi_1(z) : \quad , \tag{16}$$

from which one can obtain $c = 1/2$, and the anomalous scale dimensions of the conformal fields should be $\Delta = 0$, $\frac{1}{16}$, and $\frac{1}{2}$, which are related to the critical exponents of the Ising model[7,8].

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