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Quantum Binary Symmetric Channels

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Quantum binary symmetric channels are defined via the invariance of fidelity under unitary transformations of the input density operators. In this definition, they not only include the most studied case of the depolarizing channel but also other channels. We investigate the character of the latter and find the maximum of the coherent information to estimate the capacities of the channels.

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In recent years, considerable progress has been made in developing a general quantum theory of information and communication.[1−3] Particularly, as in classical information theory, the efficiency of information transmission is one of the most basic questions in quantum information theory. Unlike classical theory, in quantum theory there are two very different forms of transmission efficiencies, classical capacity and quantum capacity of quantum channel, depending on the nature of the information to be conveyed. Following the proof of the quantum analogue of Shannon's fundamental theorem for noiseless coding by Schumacher,[4] the problem of classical capacity of quantum channels has been solved.[5,6] In the past few years, the concept of channel fidelity has been developed[7] and a substantial amount of work has been devoted to the transmission of quantum information through noisy quantum channels.[8−10] The concept of coherent (quantum) information[8] was defined to characterize the amount of quantum information transmitted by quantum channels. It is generally believed that the maximum of coherent information represents a lower bound on quantum channel capacity in Shannon's definition. The upper bound is given by the corresponding asymptotic quantity and is very difficult to carry out for the channels other than quantum erasure. In this letter, we will calculate the maximum of the coherent information as an estimation of the channel capacity.

In what follows, we shall outline the question of how to describe the quantum channel, and introduce the symmetric property of the binary channel, then calculate the maximum of the coherent information of the quantum binary symmetric channel (QBSC). Finally, we shall give a discussion and conclusion.

Suppose a quantum system Q is subjected to a dynamical evolution, which may represent the transmission of Q via a noisy quantum channel. In general, the evolution of Q will be represented by a superoperator ε, which gives the mapping from initial states (described by density operators ρ to final states $\rho' = \varepsilon(\rho)$, where we use primes to denote states after evolution. The superoperator has a normalized operator-sum representation,[8] which involves a collection of operators A_μ acting on Q's Hilbert space $\mathcal{H}_Q$, that is

$$\rho' = \varepsilon(\rho) = \sum_\mu A_\mu \rho A_\mu^+. \tag{1}$$

The operators A_μ must satisfy a normalization condition

$$\sum_\mu A_\mu^+ A_\mu = 1^Q. \tag{2}$$

An important quantity in quantum information theory is the entropy exchange $S_{\rm e}$, which is $S_{\rm e} = S(W)$, with von Neumann entropy $S(W) = -{\rm Tr}(W \log_2 W)$, and W is a density operator with components $W_{\mu\nu} = {\rm Tr} A_\mu \rho A_\nu^+$. The coherent information is defined as

$$I_{\rm e} = S(\rho') - S_{\rm e}. \tag{3}$$

Suppose the collection of operators O_ν forming the basis for any operator acting on $\mathcal{H}_Q$ can be expanded on. Then

$$A_\mu = \sum_v A_{\mu\nu} O_\nu, \tag{4}$$

where $A_{\mu\nu}$ are the expanding coefficients. Then Eq. (1) can be expressed as

$$\rho' = \sum_{\nu\tau} B_{\nu\tau} O_\nu \rho O_\tau^+, \tag{5}$$

where

$$B_{\nu\tau} = \sum_\mu A_{\mu\nu} A_{\mu\tau}^*$$

are the elements of the channel matrix, which characterize the quantum channel. The eigenvalue problem

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of W is critical in the calculation of the coherent information. Denote $\mathrm{Tr}O_\alpha \rho O_\beta$ as $T_{\alpha\beta}$, then

$$W_{\mu\nu} = \sum_{\alpha\beta} A_{\mu\alpha} T_{\alpha\beta} A^*_{\nu\beta}.$$

In the matrix form $W = ATA^+$ and $B = A^+A$. The eigenequation of W is $|ATA^+ - \lambda| = 0$. This is equivalent to

$$|BT - \lambda| = 0, \tag{6}$$

provided that A is a square matrix and B is not singular. Accordingly for given channel matrix B and initial state ρ, the coherent information I_e can be calculated.

Firstly, let us in retrospect look at the definition of the symmetry of a classical binary channel. Such a channel is characterized by the set $X = \{0,1\}$ of possible inputs, the set of $Y = \{0,1\}$ of possible outputs, and a set of conditional probabilities that relate the possible outputs to the possible inputs. If the conditional probabilities have the properties that $P(Y = 0|X = 1) = P(Y = 1|X = 0) = p$ and $P(Y = 1|X = 1) = P(Y = 0|X = 0) = 1 - p$. Then the channel is a binary symmetric channel. We can see that symmetry means that error probability is unrelated to the inputs.

Now, in quantum theory, the inputs and outputs are quantum states of a discrete-state quantum system. Although the quantum extension of conditional probability is proposed,[9] it is not fruitful now. On the other hand, the idea of fidelity is rather commonly mentioned in the literature on quantum information. In the quantum version of the Fano inequality, exchange fidelity $F_\mathrm{e} = \sum_\mu (\mathrm{Tr}\rho A_\mu)\left(\mathrm{Tr}\rho A^+_\mu\right)$ was used, and 1-F_e roughly corresponds to the classical probability of error P_e.[8] Thus we shall make use of fidelity rather than conditional probability in investigating the symmetry of a quantum channel.

Definition: A quantum channel is strong symmetry if its exchange fidelity is invariant under any unitary transformation of the input density operators.

This is

$$\begin{aligned}&\sum_\mu \left(\mathrm{Tr}U\rho U^+ A_\mu\right)\left(\mathrm{Tr}U\rho U^+ A^+_\mu\right)\\ &= \sum_\mu (\mathrm{Tr}\rho A_\mu)\left(Tr\rho A^+_\mu\right),\end{aligned} \tag{7}$$

where U is an arbitrary unitary operator acting on $\mathcal{H}_Q$.

For the binary channel, symmetry also means strong symmetry. A particularly useful operator basis is the collection of Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ together with identity matrix I, which is widely used in quantum error-correcting coding. We denote these as $\sigma_0 = I$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$, respectively. In this operator basis, the input density operator can be expressed as

$$\rho = \frac{1}{2}\left(I + \boldsymbol{b}\cdot\boldsymbol{\sigma}\right) = \frac{1}{2}\sum_{\alpha=0}^{3} b_\alpha \sigma_\alpha,$$

$\boldsymbol{b}$ is the Bloch vector, $b = |\boldsymbol{b}| \leqslant 1$, $b_0 = 1$. Since

$$\mathrm{Tr}\left(\rho A_\mu\right) = \sum_\alpha A_{\mu\alpha}\mathrm{Tr}\left(\rho\sigma_\alpha\right) = \sum_\alpha A_{\mu\alpha} b_\alpha,$$

substituting it into Eq. (7), we conclude

$$\sum_{\alpha\beta} B_{\alpha\beta}\left(b^U_\alpha b^U_\beta - b_\alpha b_\beta\right) = 0, \tag{8}$$

where $\boldsymbol{b}^U$ is the Bloch vector of density operator $\rho^U = U\rho U^+$, and $b^U = b$ because U is a unitary operator. The solution of Eq. (8) is

$$\begin{aligned}&B_{\alpha\beta} + B_{\beta\alpha} = 0, \quad \text{for} \quad \alpha \neq \beta,\\ &B_{11} = B_{22} = B_{33}.\end{aligned} \tag{9}$$

Then we can write Eq. (2) in the form of

$$\sum_{\alpha\beta} B_{\beta\alpha}\sigma_\alpha\sigma_\beta = I,$$

combine this with Eq. (9) and obtain

$$B_{ij} = 0 \quad (i,j = 1,2,3: \quad i \neq j), \quad B_{00} + 3B_{11} = 1.$$

From the definition of B we know that B is a Hermite matrix (an important property for measurable physical quantity), so that the diagonal elements are real, and the remaining nonzero off-diagonal elements are imaginary. At last the matrix of a QBSC will be

$$B = \begin{bmatrix} p & \mathrm{i}\omega_1 & \mathrm{i}\omega_2 & \mathrm{i}\omega_3 \\ -\mathrm{i}\omega_1 & q & 0 & 0 \\ -\mathrm{i}\omega_2 & 0 & q & 0 \\ -\mathrm{i}\omega_3 & 0 & 0 & q \end{bmatrix}, \tag{10}$$

with

$$0 \leqslant q \leqslant \frac{1}{3}, \quad p + 3q = 1,$$

$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ being a real vector, and $\omega = |\boldsymbol{\omega}|$.

The explicit form of matrix T will be

$$T = \begin{bmatrix} 1 & b_1 & b_2 & b_3 \\ b_1 & 1 & -\mathrm{i}b_3 & \mathrm{i}b_2 \\ b_2 & \mathrm{i}b_3 & 1 & -\mathrm{i}b_1 \\ b_3 & -\mathrm{i}b_2 & \mathrm{i}b_1 & 1 \end{bmatrix}, \tag{11}$$

and Eq. (6) will be

$$\lambda^4 - \lambda^3 + a_2\lambda^2 - a_3\lambda + a_4 = 0, \tag{12}$$

with

$$\begin{aligned}a_2 &= (p+q)\left(3 - b^2\right)q + (\boldsymbol{\omega}\cdot\boldsymbol{b})^2 - \omega^2,\\ a_3 &= \left[q^2(3p+q) - 2q\omega^2\right]\left(1 - b^2\right),\\ a_4 &= \left(pq - \omega^2\right)q^2\left(1 - b^2\right)^2.\end{aligned}$$

Now let us consider how to solve Eq. (12) analytically in some special cases. The first case of interest is $b = 1$. Eq. (12) reduces to

$$\lambda^2 \left[\lambda^2 - \lambda + 2q\,(1-2q) - \omega^2 \sin^2\theta\right] = 0, \qquad (13)$$

where θ is the angle between $\boldsymbol{\omega}$ and $\boldsymbol{b}$. In this case, the transmitted initial state is a pure state, the coherent information will be always zero. Secondly, when $(\boldsymbol{\omega}\cdot\boldsymbol{b})^2 = \omega^2 b^2$, which includes the depolarizing channel as a special case of $\omega = 0$, Eq. (12) can be decomposed into

$$\lambda^2 - (p+q)\,\lambda + \left(1-b^2\right)\left(pq-\omega^2\right) = 0, \qquad (14)$$

$$\lambda^2 - 2q\lambda + \left(1-b^2\right)q^2 = 0. \qquad (15)$$

Then the exchange information can be expressed analytically, as can the coherent information. The density operators of the final state can now be expressed as

$$\rho' = \sum_{\alpha\beta} B_{\alpha\beta}\sigma_\alpha \rho \sigma_\beta = \frac{1}{2}\left(I + \boldsymbol{b}'\cdot\boldsymbol{\sigma}\right). \qquad (16)$$

For our symmetric channel, using Eq.(11), we have

$$\boldsymbol{b}' = (p-q)\,\boldsymbol{b} + 2\boldsymbol{\omega}\times\boldsymbol{b}. \qquad (17)$$

The eigenvalues of the density operator of final state are $\lambda'_{\pm} = (1 \pm b')/2$, with $b' = \left|\boldsymbol{b}'\right|$. The von Neumann entropy of the final state is

$$S(\rho') = -\lambda'_+ \log_2 \lambda'_+ - \lambda'_- \log_2 \lambda'_-. \qquad (18)$$

By numerical calculation we know that the maximum of the coherent information of QBSC appears at the $b = 0$ point or $b = 1$ sphere. For the general situation, although the coherent information cannot be easily described analytically in an explicit form, fortunately its maximum can be denoted by $C_{\rm es}$ as an estimation of channel capacity. When it is positive (otherwise it is zero),

$$C_{\rm es} = 1 + 2q\log_2 q + \lambda_1 \log_2 \lambda_1 + \lambda_2 \log_2 \lambda_2, \qquad (19)$$

with

$$\lambda_{1,2} = \frac{1}{2}\left(1 - 2q \pm \sqrt{(1-4q)^2 + 4\omega^2}\right).$$

We now give an interpretation of the definition of symmetry of quantum channel. Fidelity is a general way of defining the "closeness" of a pair of states. Exchange fidelity or entanglement fidelity[8] measures how faithfully the pure entangled state (a state of Q entangled with a "reference system") is preserved by the dynamics of Q. The transformation invariance of fidelity is somewhat sophisticated, however, a simple example with physical reality is available. Let us consider an optical fibre and polarized photon states transmitted on it; for simplicity let the transmission direction be strictly along the axis of the fibre. It is reasonable to assume that the fibre is of rotational symmetry. Then if the plane of polarization of the initial state is rotated with a certain angle, the final state at the other side of the fibre will rotate with the same angle. The difference of the two states will be unrelated to the plane of polarization the sender chooses. Therefore, by a rotation of initial state the fidelity of transmission is invariant.

In conclusion, strong symmetry of quantum channel is defined in a reasonable way. A channel matrix is obtained for QBSC and the analytical expression of the maximum of coherent information is obtained as an estimation of the quantum capacity of the rotating and depolarizing channel.

References

[1] Bennett C H and Shor P W 1998 *IEEE Trans. Inf. Theory* **44** 2724

[2] Yang C P and Guo G C 1999 *Chin. Phys. Lett.* **16** 628

[3] Zheng S B and Zhu X W 2000 *Chin. Phys. Lett.* **17** 391

[4] Schumacher B 1995 *Phys. Rev.* A **51** 2738

[5] Hausladen P et al 1996 *Phys. Rev.* A **54** 1869

[6] Holevo A S, Sohma M and Hirota O 1999 *Phys. Rev.* A **59** 1820

[7] Jozsa R 1995*J. Mod. Opt.* **41** 2315

[8] Schumacher B 1996 *Phys. Rev.* **54** 2614

Schumacher B and Nielsen M A 1996 *Phys. Rev.* A **54** 2629

[9] Cerf N J and Adami C 1999 *Phys. Rev.* A **60** 893

Cerf N J 1998 *Phys. Rev.* A **57** 3330

[10] Lloyd S 1997 *Phys. Rev.* A **55** 1613