Quantum Multiple Access Channel *

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(Received 13 August 2001)

We consider the transmission of classical information over a quantum channel by many senders, which is a generalization of the two-sender case. The channel capacity region is shown to be a convex hull bound by the von Neumann entropy and the conditional von Neumann entropies. The result allows a reasonable distribution of channel capacity over the senders.

PACS: 03. 65. Ca, 03. 80. +r, 03. 65. Ge, 02. 30. Mv

The theory of information transformation over quantum channels has been widely discussed.[1−10] In this letter, we consider the transmission of classical information by N senders to a common receiver, which is the generalization of the two-sender scheme.[11] We suppose N senders, $S_1, S_2, \cdots, S_N$, are given an ensemble of letter states $|\alpha_1\alpha_2\cdots\alpha_N\rangle$, where $\alpha_1, \alpha_2, \cdots, \alpha_N$ can be drawn from N alphabet sets: $\mathcal{H}_1 = \{\alpha_1\}, \cdots, \mathcal{H}_N = \{\alpha_N\}$. While S_1 is allowed to choose the letter $\alpha_1, \cdots$, S_N is allowed to choose the letter α_N. Then the letter state is sent to the receiver, Roger, who subjects it to a measurement to determine which letters the senders have chosen. Provided that $p_{\alpha_1}\cdots p_{\alpha_N}$ is a product probability distribution of the letter state $|\alpha_1\alpha_2\cdots\alpha_N\rangle$, then the density matrix can be denoted by

$$\rho = \sum_{\alpha_1\cdots\alpha_N} p_{\alpha_1}\cdots p_{\alpha_N} |\alpha_1\alpha_2\cdots\alpha_N\rangle\langle\alpha_1\alpha_2\cdots\alpha_N|,$$

$$\rho_{\alpha_{i_{m+1}}\cdots\alpha_{i_N}} = \sum_{\alpha_{i_1}\cdots\alpha_{i_m}} p_{\alpha_{i_1}}\cdots p_{\alpha_{i_m}} |\alpha_1\alpha_2\cdots\alpha_N\rangle\langle\alpha_1\alpha_2\cdots\alpha_N|.$$

We denote the conditional von Neumann entropy by

$$H_{i_1\cdots i_m} = \sum_{\alpha_{i_{m+1}}\cdots\alpha_{i_N}} p_{\alpha_{i_{m+1}}}\cdots p_{\alpha_{i_N}} H\left(\rho_{\alpha_{i_{m+1}}\cdots\alpha_{i_N}}\right), \quad (1)$$

where $(i_1, \cdots, i_m)$ is a combination chosen from $\{1, 2, \cdots, N\}$, and $(i_{m+1}, \cdots, i_N)$ is the complementary set. We easily have $H_{12\cdots N} = H(\rho)$.

We can see $H_{i_1\cdots i_m} \leq H(\rho)$ from the concavity of the von Neumann entropy.[9] A code $((M_1, \cdots, M_N), l)$ consists of M_1 α_1-letter sequences, M_2 α_2-letter sequences, $\cdots$. They form $M_1\cdots M_N$ code words $\{|W_{i_1\cdots i_N}\rangle : i_1 = 1, 2, \cdots, M_1; \cdots; i_N = 1, 2, \cdots, M_N\}$, with each α_1-letter sequence, α_2-letter sequence, $\cdots$, combined. All the sequences are of length l. We assume that the $M_1\cdots M_N$ code words have the same probability $1/M_1\cdots M_N$.

A rate $(R_1, \cdots, R_N)$ is said to be achievable if there is a sequence of $\left(\left(2^{nR_1}, \cdots, 2^{nR_N}\right), n\right)$ codes for which Roger can decode the message by a measurement with the error probability $P_E \to 0$ when n tends to infinity. The capacity region of the multiple access channel is the closure of the set of achievable $(R_1, \cdots, R_N)$ rate combinations. The following theorem is the main result of this letter.

Theorem The capacity region of a quantum multiple access channel is the closure of the convex hull of all $(R_1, \cdots, R_N)$ satisfying

$$R_{i_1} + \cdots + R_{i_m} < H_{i_1\cdots i_m} \quad (2)$$

for an arbitrary combination $(i_1, \cdots, i_m)$ chosen from $1, 2, \cdots, N$.

Thus $H_{i_1\cdots i_m}$ is the quantum analogy of classical conditional mutual information.[5] Our result provides an information-theoretical interpretation of the conditional von Neumann entropy.

We first give a lemma about the conditional von Neumann entropy, which can be proven by the strong subadditivity of the von Neumann entropy.[9]

Lemma For an arbitrary combination $(i_1, \cdots, i_n)$ chosen from $1, 2, \cdots, N$,

$$H_{i_1\cdots i_m} + H_{i_{m+1}\cdots i_n} \geq H_{i_1\cdots i_n} \quad (m < n). \quad (3)$$

Another property of the quantum multiple access channel is its convexity, i.e. if $(R_1, \cdots, R_N)$ and $\left(R_1', \cdots, R_N'\right)$ are achievable rates, then

$$\left(\lambda R_1 + (1-\lambda)R_1', \cdots, \lambda R_N + (1-\lambda)R_N'\right)$$

is also achievable, for any $0 \leq \lambda \leq 1$. This idea is the time sharing scheme. Given two sequences of codes at different rates $(R_1, \cdots, R_N)$ and $\left(R_1', \cdots, R_N'\right)$, we can construct a third code book at the rate of

*Supported by the National Natural Science Foundation of China under Grant No. 19975043.
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$\lambda(R_1,\cdots,R_N)+(1-\lambda)\left(R_1^{'},\cdots,R_N^{'}\right)$ by using the first code book for the first λn symbols and the second for the remaining $(1-\lambda)n$ symbols. Since the overall error probability is less than the sum of the error probability for each of the segments, the error probability of the new code approaches zero; the rate is achievable.

According to Eq.(3), all $(R_1,\cdots,R_N)$ satisfying Eq.(2) form a convex polyhedron, which has the following $N!$ outermost vertices (besides the trivial vertex $(0,\cdots,0)$):

$$\begin{aligned}&\text{Serial Number}\rightarrow \textit{Polyhedron Vertex}\\&(1,\cdots,N)\rightarrow(H_1,H_{12}-H_1,H_{123}-H_{12},\cdots,\\&\qquad H_{12\cdots N}-H_{12\cdots(N-1)})\\&\qquad\cdots\\&(N,\cdots,1)\rightarrow(H_{12\cdots N}-H_{23\cdots N},\cdots,\\&\qquad H_{(N-1)N}-H_N,H_N).\end{aligned}$$

In other words, for the permutation $(i_1,\cdots,i_N)$, the rate of the corresponding vertex reads

$$\begin{aligned}R_{i_1}&=H_{i_1},R_{i_2}=H_{i_1i_2}-H_{i_1},\cdots,\\R_{i_N}&=H_{i_1i_2\cdots i_N}-H_{i_1i_2\cdots i_{(N-1)}}.\end{aligned}$$

According to the convexity of the capacity region, in order to prove that all the rate combinations satisfying Eq. (2) are achievable, we need only to prove that the vertices are achievable. In fact, we only need to prove that one particular vertex $\left(H_1,H_{12}-H_1,H_{123}-H_{12},\cdots,H_{12\cdots N}-H_{12\cdots(N-1)}\right)$ is achievable. This can be done in the same way as in the two-sender scheme.[11]

We denote $\mathfrak{E}$ as the closure of the convex hull of all $(R_1,\cdots,R_N)$ satisfying Eq.(2). Supposing the senders can send information to Roger at the rate of $(R_1,\cdots,R_N)$, then we prove that $(R_1,\cdots,R_N)\in\mathfrak{E}$. This is just the converse theorem. We consider a $\left(\left(2^{lR_1},\cdots,2^{lR_N}\right),l\right)$ code in which the code words are $|W_{a_1\cdots a_N}\rangle$, where a_i is S_i signal string (of length l). Supposing Roger can decode the signals asymptotically error-free, then when l is sufficiently large we have the inequality

$$R_{i_1}+\cdots+R_{i_m}\leq\frac{1}{l}I\left(\text{Roger: }S_{i_1},\cdots,S_{i_m}|S_{i_{m+1}},\cdots,S_{i_N}\right)\tag{4}$$

Letting $M_i=2^{lR_i}$, we denote

$$\rho_{\text{code}}^{a_{i_{m+1}}\cdots a_{i_N}}=\sum_{a_{i_1}\cdots a_{i_m}}\frac{1}{M_{i_1}\cdots M_{i_m}}|W_{a_1\cdots a_N}\rangle\langle W_{a_1\cdots a_N}|$$

and their von Neumann entropies by $H_{code}^{a_{i_1}\cdots a_{i_m}}$. According to the Holevo theorem,[3] the mutual information is bound by the von Neumann entropies

$$\begin{aligned}&I\left(\text{Roger: }S_{i_1},\cdots,S_{i_m}|S_{i_{m+1}},\cdots,S_{i_N}\right)\\&\leq\sum_{a_{i_{m+1}}\cdots a_{i_N}}\frac{1}{M_{i_{m+1}}\cdots M_{i_N}}H\left(\rho_{\text{code}}^{a_{i_{m+1}}\cdots a_{i_N}}\right).\end{aligned}\tag{5}$$

Letting $\mathcal{E}$ be the ensemble of letter states that appear as the first letters in the code words, we then have a product distribution, and we can define the conditional entropies $H_{i_1\cdots i_m}^{(1)}$. Similarly we define $H_{i_1\cdots i_m}^{(k)}$ for each position $k=1,2,\cdots,l$ in the code words. According to the subadditivity of the von Neumann entropy,[9] we have

$$\begin{aligned}&\sum_{a_{i_{m+1}}\cdots a_{i_N}}\frac{1}{M_{i_{m+1}}\cdots M_{i_N}}H\left(\rho_{code}^{a_{i_{m+1}}\cdots a_{i_N}}\right)\\&\leq H_{i_1\cdots i_m}^{(1)}+\cdots+H_{i_1\cdots i_m}^{(l)}.\end{aligned}\tag{6}$$

By combining Eqs. (4)−(6), we have

$$R_{i_1}+\cdots+R_{i_m}\leq\frac{1}{l}\left(H_{i_1\cdots i_m}^{(1)}+\cdots+H_{i_1\cdots i_m}^{(l)}\right).\tag{7}$$

We denote

$$H_{i_1\cdots i_m}=\frac{1}{l}\sum_{k=1}^{l}H_{i_1\cdots i_m}^{(k)}.$$

Since

$$\left(H_1^{(k)},H_{12}^{(k)}-H_1^{(k)},\cdots,H_{12\cdots N}^{(k)}-H_{12\cdots(N-1)}^{(k)}\right),\ldots\in\mathfrak{E},\qquad k=1,2,\cdots,l,$$

and considering the convexity of $\mathfrak{E}$, we know that $\left(H_1,H_{12}-H_1,\cdots,H_{12\cdots N}-H_{12\cdots(N-1)}\right),\ldots$ are also in $\mathfrak{E}$. According to Eq.(3), all $(R_1,\cdots,R_N)$ satisfying Eq.(7) form a convex polyhedron, of which $\left(H_1,H_{12}-H_1,\cdots,H_{12\cdots N}-H_{12\cdots(N-1)}\right),\ldots$ are the $N!$ outermost vertices. It follows that it must be $(R_1,\cdots,R_N)\in\mathfrak{E}$. Thus we complete the proof of the converse of the theorem.

The above theorem provides some intriguing quantum communication schemes, which can be viewed as a generalized superdense coding scheme. The superdense coding scheme proposed by Bennett and Wiesner[10] dealt with two-partite communication, but here we deal with $(N+1)$-partite communication. We suppose that the senders want to send classical information to Roger by M-state quantum systems; if they send the message independently, they can send $\log_2 M$ bits per system. However, we suppose that they initially share a considerable supply of M-state entanglement, so how can they expand their channel capacity? Note that the senders may be at different locations, so each of them must encode the messages independently of the others by a predetermined code.

We suppose that the initial state the senders share is $\rho_0=|\psi\rangle\langle\psi|$. Then the density matrix for

$S_{i_1}, \cdots, S_{i_m}$ is $\rho^{i_1 \cdots i_m} = tr_{i_{m+1} \cdots i_N}(\rho_0)$. We denote $H_E^{i_1 \cdots i_m} = S\left(\rho^{i_1 \cdots i_m}\right)$, which can be used to measure the entanglement between systems. Each of the senders can perform a unitary transformation on his system, and then convey the system to Roger, respectively. Supposing they can send information at a rate $(R_1, \cdots, R_N)$, then according to our theorem, it must be

$$R_1 + \cdots + R_N \leq N \log_2 M,$$

$$R_{i_{m+1}} + \cdots + R_{i_N} \leq (N-m) \log_2 M + H_E^{i_1 \cdots i_m}. \quad (8)$$

Note that all $(R_1, \cdots, R_N)$ satisfying Eq.(8) can be achieved with the transformation ensembles of the senders, including all the permutations of a chosen basis state of the initial state $|\psi\rangle$, rotations of the relative phases of these states, and the combination of the two cases.

This is useful because the amount of information the senders want to send to Roger may be different, if S_i has more information to send, we can adopt a code that increases the channel capacity of S_i at the sacrifice of the others. This then allows us to distribute the channel capacity between users properly, without wasting the entanglement.

The result in this letter has opened the way to exploit the classical capacity of some multi-partite communication schemes over quantum channels. This method will be useful in future research to study some more general network communications over quantum channels. In network communication, the users can send and receive information in two-way channels. The situation is so complicated that even in the classical case the channel capacity cannot be calculated exactly. Now we expect that some simple cases may be generalized to quantum channels.

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