CHIN.PHYS.LETT. Vol. 19, No. 4 (2002) 515

# Lattice Bhatnagar–Gross–Krook Simulations of Turbulent Natural Convection in a Cavity *

SHI Bao-Chang (施保昌)[1], GUO Zhao-Li (郭照立)[2], WANG Neng-Chao (王能超)[1]
[1]*Parallel Computing Institute, Huazhong University of Science and Technology, Wuhan 430074*
[2]*National Laboratory of Coal Combustion, Huazhong University of Science and Technology, Wuhan 430074*

(Received 18 September 2001)

*A thermal lattice Bhatnagar–Gross–Krook (LBGK) model with a robust boundary scheme is developed for Boussinesq incompressible fluids. In the model the velocity and temperature fields are solved by two independent LBGK equations which are combined into a coupled equation for the whole system. The two-dimensional natural convection flow of air in a differentially heated cavity of an aspect ratio of four is simulated for the Rayleigh number up to $10^{10}$. The numerical results are found to be in good agreement with those of previous studies.*

*PACS: 47.11.+j, 44.25.+f*

Natural convection flow in a differentially heated cavity is a classical problem in the literature on heat transfer and it serves as one of the most popular test-problems for numerical methods for incompressible recirculating viscous flows. In the past years studies relevant to natural convection in cavities have mainly focused on laminar cases for values of the Rayleigh number ($R_a$) below $10^8$, and benchmark solutions have been obtained.[1,2] Le Quéré *et al.* performed detailed numerical simulations of the two-dimensional natural convection in cavities with adiabatic horizontal walls for values of $R_a$ up to $10^{10}$ using a pseudospectral scheme.[2−4] Due to the complex behaviour of the natural convection flows at large $R_a$, the solution of these flows is very difficult. New methods with high performance are still needed for further study, especially for three-dimensional problems. The lattice Bhatnagar–Gross–Krook (LBGK) method is a candidate for such algorithms.[5]

In recent years, the LBGK method has achieved great success in simulations of fluid flows and modelling physics in fluids.[5−17] Compared with the conventional computational fluid dynamics approach, the LBGK method is simple to code, intrinsically parallel, and ready for extending to three-dimensional problems. It is also easy to incorporate complicated boundary conditions such as those in porous media. Previously, the LBGK method has mainly been used for isothermal flows. Recently, some thermal LBGK (TLBGK) models have been developed.[6−9] However, most of these models suffer severely from numerical instability,[10] and the evolution equations are quite complicated.

In this Letter, a new TLBGK model with a robust boundary scheme is developed for Boussinesq incompressible fluids by extending our previous isothermal incompressible LBGK model.[11] The TLBGK model is similar to those in Refs. [7] and [8], but is simpler and more flexible because different lattices can be used. Using the model, we simulated the two-dimensional natural convection in a differentially heated air-filled cavity of an aspect ratio of four with adiabatic horizontal walls for values of $R_a$ up to $10^{10}$. The numerical results are found to be in good agreement with those of previous studies.

First, let us describe the problem. The cavity considered is of height $H$ and width $W$ (aspect ratio $A = H/W$). The flow in the cavity is governed by two-dimensional unsteady Boussinesq equations in primitive variables, and the dimensionless equations for the flow read[2−4]

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \upsilon \nabla^2 \boldsymbol{u} + P_r \Theta \boldsymbol{k}, \tag{1b}$$

$$\frac{\partial \Theta}{\partial t} + \boldsymbol{u} \cdot \nabla \Theta = D \nabla^2 \Theta, \tag{1c}$$

where $\boldsymbol{u} = (u, v)$ is the velocity vector, $\boldsymbol{k}$ is the unit vector in the $y$-direction, $p$ and $\Theta$ are the pressure and temperature, $\upsilon = P_r R_a^{-0.5}$, $P_r$ is the Prandtl number and $D = R_a^{-0.5}$.

The boundary conditions are taken to be $u = v = 0$ on all four walls; $\Theta = 0.5$ on $x = 0$ and $\Theta = -0.5$ on $x = 1/A, 0 \leqslant y \leqslant 1$; $\partial\Theta/\partial y = 0$ on $y = 0$ and 1, $0 \leqslant x \leqslant 1/A$. Here we set $P_r = 0.71$ (air) and $A = 4$.

For the thermal LBGK model, we first modify the LBGK model described in Ref. [11] for the velocity field by adding a term to the evolution equation

$$\begin{aligned} & g_i\left(\boldsymbol{x} + c\boldsymbol{e}_i \Delta t, \quad t + \Delta t\right) - g_i\left(\boldsymbol{x}, t\right) \\ & = -\tau^{-1}\left[g_i\left(\boldsymbol{x}, t\right) - g_i^{(0)}\left(\boldsymbol{x}, t\right)\right] + f_i, \end{aligned} \tag{2}$$

where $\boldsymbol{e}_i$ are the discrete velocity directions defined by

$$\boldsymbol{e}_0 = 0,$$
$$\boldsymbol{e}_i = \left(\cos\left[(i-1)\pi/2\right], \sin\left[(i-1)\pi/2\right]\right) \quad \text{for} \quad i = 1:4,$$

*Supported by the National Natural Science Foundation of China under Grant No 60073044 and the Special Fund for Major State Basic Research Project of China under Grant No G1999022207.
©2002 Chinese Physical Society and IOP Publishing Ltd

$$\boldsymbol{e}_i = \sqrt{2}\left(\cos\left[(i-5)\,\pi/2+\pi/4\right],\right.$$
$$\left.\sin\left[(i-5)\,\pi/2+\pi/4\right]\right) \quad \text{for} \quad i=5:8.$$

$c=\Delta x/\Delta t$ is the particle speed satisfying incompressible limitation $|\boldsymbol{u}| \ll c$, $g_i(\boldsymbol{x},t)$ is the distribution function, and $g_i^{(0)}(\boldsymbol{x},t)$ is the equilibrium defined by

$$g_i^{(0)} = \lambda_i p + s_i(\boldsymbol{u}), \tag{3}$$

where $\lambda_0 = -4\sigma/c^2$, $\lambda_i = -\lambda/c^2$ $(i=1:4)$ and $\lambda_i = -\gamma/c^2$ $(i=5:8)$ with parameters $\sigma,\lambda$, and $\gamma$ satisfying $\lambda+\gamma=\sigma$, $\lambda+2\gamma=1/2$, and

$$s_i(\boldsymbol{u}) = \omega_i\left[3\frac{(\boldsymbol{e}_i\cdot\boldsymbol{u})}{c} + 4.5\frac{(\boldsymbol{e}_i\cdot\boldsymbol{u})^2}{c^2} - 1.5\frac{|\boldsymbol{u}|^2}{c^2}\right]$$

with the weights $\omega_0=4/9$, $\omega_i=1/9\,(i=1:4)$, and $\omega_i=1/36 \quad (i=5:8)$. Here we take

$$f_i = \frac{P_r}{2c}\Delta t\alpha_i \boldsymbol{e}_i\cdot\boldsymbol{k}\Theta$$

in Eq. (2), with $\alpha_i=\delta_{i2}+\delta_{i4}$. Other forms of $f_i$ can also be used.[12]

The flow velocity, pressure and kinetic viscosity are given by

$$\boldsymbol{u} = \sum_{i=1}^{8} c\boldsymbol{e}_i g_i, \quad p = \frac{c^2}{4\sigma}\left[\sum_{i=1}^{8} g_i + s_0(\boldsymbol{u})\right],$$
$$v = \frac{(2\tau-1)}{6}\frac{(\Delta x)^2}{\Delta t}. \tag{4}$$

To solve Eq. (1c), we use the LBGK equation proposed in Ref. [13], which utilized a lattice consisting of $\boldsymbol{e}_i$ $(i=1:4)$ as used in Eq. (2). It should be noted that the other lattices could also be used. However, the lattice introduced here is the simplest one. The LBGK equation for Eq. (1c) reads

$$\Theta_i(\boldsymbol{x}+c\boldsymbol{e}_i\Delta t, \quad t+\Delta t) - \Theta_i(\boldsymbol{x},t)$$
$$= -\tau_\Theta^{-1}\left[\Theta_i(\boldsymbol{x},t) - \Theta_i^{(0)}(\boldsymbol{x},t)\right], \tag{5}$$

where $\Theta_i^{(0)} = (\Theta/4)\left[1+2\boldsymbol{e}_i\cdot\boldsymbol{u}/c\right]$ is the equilibrium. $\Theta$ and $D$ are calculated by

$$\Theta = \sum_{i=1}^{4}\Theta_i, \quad D = \frac{(2\tau_\Theta-1)}{4}\frac{(\Delta x)^2}{\Delta t}. \tag{6}$$

Through multiscaling expansion, Eqs. (1a) and (1b) can be derived from Eq. (2) with Eq. (4) to the order of $O(\Delta t^2)$, while Eq. (1c) can be derived from the LBGK Eq. (5) with Eq. (6) to the same order if the Mach number $M=O(\Delta t)$. Thus a coupled TLBGK model is obtained for problem (1).

For the boundary conditions, we have found that most of the existing boundary schemes commonly used in LBGK usually encounter numerical instability for flows at large $R_a$ or $R_e$. In a previous study we developed a non-equilibrium extrapolation rule for velocity and pressure boundary conditions,[14] which is of second order, simple form, and exhibits much better numerical stability. In this study we use this scheme for the velocity boundary, which is given by

$$g_i(\boldsymbol{x}_b,t) = \bar{g}_i^{(0)}(\boldsymbol{x}_b,t) + \left(g_i(\boldsymbol{x}_f,t) - g_i^{(0)}(\boldsymbol{x}_f,t)\right), \tag{7}$$

where $\bar{g}_i^{(0)}(\boldsymbol{x}_b,t) = \lambda_i p(\boldsymbol{x}_f,t) + s_i(\boldsymbol{u}(\boldsymbol{x}_b,t))$, $\boldsymbol{x}_b$ is a node on the boundary, and $\boldsymbol{x}_f$ is its nearest-neighbour fluid node.

Similarly, for the thermal boundary condition, we use

$$\Theta_i(\boldsymbol{x}_b,t) = \begin{cases} \Theta_i^0(\boldsymbol{x}_b,t) + \left(\Theta_i(\boldsymbol{x}_f,t) - \Theta_i^0(\boldsymbol{x}_f,t)\right), \\ \qquad \text{if} \quad \Theta(\boldsymbol{x}_b,t) \quad \text{is} \quad \text{known}, \\ \frac{\Theta(\boldsymbol{x}_f,t)}{4}\left[1+2\frac{\boldsymbol{e}_i\cdot\boldsymbol{u}(\boldsymbol{x}_b,t)}{c}\right] \\ +\left(\Theta_i(\boldsymbol{x}_f,t) - \Theta_i^0(\boldsymbol{x}_f,t)\right) \quad \text{otherwise}, \end{cases} \tag{8}$$

It can be easily shown that the boundary conditions Eq. (8) for problem (1) are both of second order.

In the simulations, $(\sigma,\lambda,\gamma)$ in Eq. (3) is set to be $(5/12, 1/3, 1/12)$ such that $\lambda:\gamma=\omega_1:\omega_5$. We find that the simulations with this setting are quite robust. We simulate the flow with $R_a$=$10^7$, $10^8$, $6.4\times10^8$, $2\times10^9$, and $10^{10}$, respectively, based on a $512\times128$ uniform lattice. The corresponding relaxation parameter $\omega=\tau^{-1}$ is set to be 1.50, 1.75, 1.90, 1.94 and 1.972, respectively, and $\tau_\Theta$ is given by Eq. (6). The simulations for higher $R_a$ start from the solution obtained for lower $R_a$ as the initial condition. In the simulations, steady solutions for $R_a$=$10^7$ and $10^8$ are reached. For $R_a=6.4\times10^8$, $2\times10^9$ and $10^{10}$, the flow becomes unsteady, and time-averaged solutions are obtained over $2\times10^5$, $2.5\times10^5$ and $3.5\times10^5$ time steps, respectively. All simulations were performed on a Pentium 4 (1.4G) microcomputer, and the computational cost of 25 000 iterations required about one hour of CPU time. It is well known that the numerical stability of the LBGK model is usually very poor as $\omega=\tau^{-1}\to2$ $[\omega\in(0,2)$ from Eq. (4)]. However we found that our scheme was still stable and accurate even when $\omega=1.985$, at which the computation blows up using other schemes for the velocity and temperature boundary conditions. The streamlines and isotherms of the flows are shown in Fig. 1, and vertical temperature profiles at mid-width are plotted in Fig. 2. From the figures, we can see that the main features of the flows at $R_a=6.4\times10^8$, $2\times10^9$, and $10^{10}$ are in agreement with those obtained in Ref. [3]. To quantify the results, the average Nusselt numbers ($N_{\text{uave}}$) obtained by the TLBGK along the hot wall, as well as those in Ref. [3], are listed in Table 1. It is obvious that the two solutions are in good agreement.

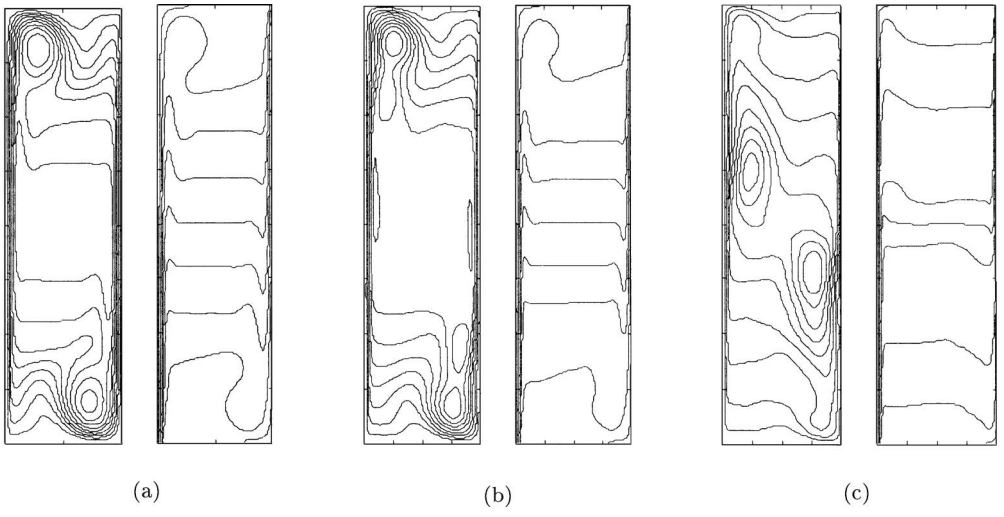



**Fig. 1.** Time-averaged solutions: (a) $R_a = 6.4 \times 10^8$; (b) $R_a = 2 \times 10^9$; (c) $R_a = 10^{10}$. For each solution the streamlines are on the left and the isotherms are on the right.

The relative errors of $N_{\text{uave}}$ between the TLBGK solutions and those in Ref. [3] are less than 3.51% for $R_a = 6.4 \times 10^8$, $2 \times 10^9$ and $10^{10}$ (see the last column in Table 1.).

Table 1. Simulation results of the natural convection flow.

| $R_a$ | | $T_{\text{step}}$ | $N_{\text{uave}}$ | $R_{\text{error}}$ |
|---|---|---|---|---|
| $10^7$ | a | 64000 | 17.3422 | |
| $10^8$ | a | 100000 | 30.6676 | |
| $6.4 \times 10^8$ | a | 200000 | 48.4589 | 0.0151 |
| | b | – | 49.2 | |
| $2 \times 10^9$ | a | 250000 | 64.7169 | 0.0268 |
| | b | – | 66.5 | |
| $10^{10}$ | a | 350000 | 97.4507 | 0.0351 |
| | b | – | 101.0 | |

[a] Present work; [b] Ref. [3]; $T_{\text{step}}$, time steps when the solutions are obtained.

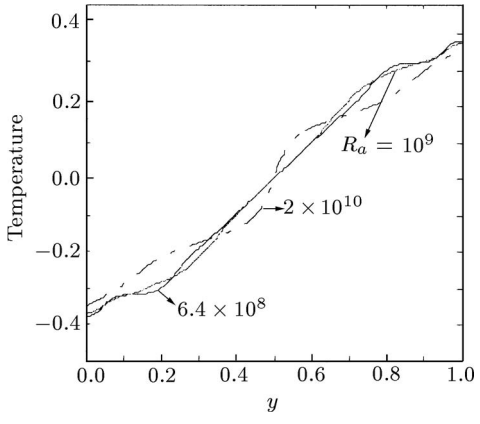



**Fig. 2.** Vertical temperature profile at mid-width.

In conclusion, we have presented a new TLBGK model for the Boussinesq incompressible flow. With the proposed boundary schemes, we can simulate the turbulent natural convection flow at large $R_a$. We found that other LBGK models with these boundary schemes also have better stability. It is also found that if a finer lattice is used, flows at $R_a > 10^{10}$ can be simulated using the present model. In fact, selecting $\omega \in (0, 2)$, $c$ ($\gg |\boldsymbol{u}|$), etc, properly in their stable regions (here $\omega \in (0, 1.985)$, $c \approx 2$) according to Eq. (4), we can simulate the flows at different $R_a$, large or small. This is just one of the advantages of the LBGK method. Moreover, our TLBGK model can be easily extended to the three-dimensional problem. Because of the complexity and very large computational costs involved for $R_a > 10^{10}$ or the three-dimensional problem, more efficient techniques and methods, such as the non-uniform lattice model and parallel technique, are needed. We shall strive to study these problems in the near future.

# References

[1] Davis G de V 1983 *Int. J. Numer. Methods Fluids* **3** 249
[2] Le Quéré P 1991 *Comput. Fluids* **20** 29
[3] Xin S and Le Quéré P 1995 *J. Fluid Mech.* **304** 87
[4] Le Quéré P and Behnia M 1998 *J. Fluid Mech.* **359** 81
[5] Chen S Y and Doolen G D 1998 *Annu. Rev. Fluid Mech.* **30** 329
[6] Chen Y, Ohashi H and Akiyama M 1994 *Phys. Rev.* E **50** 2776
[7] Shan X W 1997 *Phys. Rev.* E. **55** 2780
[8] He X Y, Chen S Y and Doolen G D 1998 *J. Comput. Phys.* **46** 282
[9] Sun C H 2000 *Chin. Phys. Lett.* **17** 209
[10] McNamara G R, Garcia A L and Alder B J 1995 *J. Stat. Phys.* **81** 395
[11] Guo Z L, Shi B C and Wang N C 2000 *J. Comput. Phys.* **165** 288
[12] Qian Y H, Succi S and Orszag S A 1995 *Ann. Rev. Comput. Phys.* **3** 195
[13] Guo Z L, Shi B C and Wang N C 1999 *J. Sci. Comput.* **14** 291
[14] Guo Z L, Zheng C G and Shi B C 2000 *Proc. Int. Conf. Appl. Comput. Fluid Dynamics* p198
[15] Yu H D and Zhao K H 1999 *Chin. Phys. Lett.* **16** 271
[16] Yan G W and Song M 1999 *Chin. Phys. Lett.* **16** 109
[17] Yan G W 2001 *Chin. Phys. Lett.* **18** 918