Analytical Results of Eigenstates and Eigenenergies by Mixing Models of Five Bosonic Modes *

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(Received 17 February 2004)

The analytical expressions of eigenstates and eigenenergies are demonstrated using a parameter λ without the assumption of Bethe ansatz for two kinds of five-mode mixing models. It is found that the parameter can be determined by the roots of a simple polynomial. We also illustrate the exact analytical expressions of eigenstates and energies without any unknown parameter.

PACS: 03. 75. Hh, 05. 30. Jp, 03. 65. Ud, 42. 50. − p

So far, nonlinear multi-wave mixing processes as well as the experimental realization of Bose–Einstein condensation (BEC) have attracted much attention.[1−3] Great efforts have been devoted to numerate the energy spectra and eigenstates of these nonlinear processes.[4−12] Usually one can research the spectra by applying the Bethe ansatz[6,7] that utilizes several Bethe parameters to give the shape of energy eigenstates. However, the reduced equations for the Bethe parameters are too complicated to be solved with analytical or numerical methods. Except for the spectra, the significance of eigenstates presents in the field of quantum information and quantum computation with the operation of states.[13] Both the spectra and the eigenstates play important roles for different complete quantized nonlinear models.

In this Letter, we expand the efficient method[14−19] to deal with five-mode mixing models. This method is effective to study the spectra and eigenstates without the assumption of the Bethe ansatz, which is very complicated and difficult to deal with.

The fully quantized Hamiltonian for the five-mode mixing models reads as

$$H = H_0 + gH_1,$$

with

$$H_0 = \sum_{j=1}^{5} \omega_1 a_j^+ a_j,$$

$$H_1 = a_1^+ a_2 a_3 a_4 a_5 + a_5^+ a_4^+ a_3^+ a_2^+ a_1, \quad \text{for } \omega_1 = \omega_2 + \omega_3 + \omega_4 + \omega_5, \tag{1a}$$

$$H_1 = a_1^+ a_2 a_3^+ a_4 a_5 + a_5^+ a_4^+ a_3 a_2^+ a_1, \quad \text{for } \omega_1 + \omega_3 = \omega_2 + \omega_4 + \omega_5, \tag{1b}$$

where ω_j and $a_j^+(a_j)$ denote the frequency and creation operator (annihilation operator) corresponding to the jth bosonic mode, and g represents the five-mode mixing coupling.

We concentrate on eigenenergies and eigenstates of the model (1a) while we summarize the results of model (1b) later. First, it should be pointed out that the six operators of free Hamiltonian H_0 and interacting Hamiltonian H_1 in Eq. (1a), $M = a_1^+ a_1 + a_2^+ a_2$, $N = a_1^+ a_1 + a_3^+ a_3$, $K = a_1^+ a_1 + a_4^+ a_4$, $L = a_1^+ a_1 + a_5^+ a_5$ commute mutually with each other and $H_0 = \omega_2 M + \omega_3 N + \omega_4 K + \omega_5 L$. As a result, M, N, K, and L are conserved quantities, and M, N, K, L and H_1 in Eq. (1a) constitute a complete set of common eigenvectors. The energy eigenvalue E can be obtained explicitly by

$$H|\psi_{M,N,K,L;\lambda}\rangle = E_{M,N,K,L;\lambda}|\psi_{M,N,K,L;\lambda}\rangle,$$
$$E = M\omega_2 + N\omega_3 + K\omega_4 + L\omega_5 + g\lambda, \tag{2a}$$
$$H_1|\psi_{M,N,K,L;\lambda}\rangle = \lambda|\psi_{M,N,K,L;\lambda}\rangle, \tag{2b}$$

where M, N, K, and L all can be taken to be $0, 1, 2 \ldots$, and

$$|\psi_{M,N,K,L;\lambda}\rangle = S(a_1^+, a_2^+, a_3^+, a_4^+, a_5^+)|\text{vac}\rangle,$$
$$S(x_1, x_2, x_3, x_4, x_5) = \sum_{j=0}^{q} \frac{\alpha_j}{j!} x_1^j x_2^{M-j} x_3^{N-j} x_4^{K-j} x_5^{L-j}, \tag{3}$$

$q = \min(K, L, M, N)$, $|\text{vac}\rangle$ represents the vacuum state, and $x_j = a_j^+$ $(j = 1, 2, 3, 4, 5)$. Since $H_1|\text{vac}\rangle = 0$, we find $H_1|\psi\rangle = H_1 S|\text{vac}\rangle = [H_1, S]|\psi\rangle$. We can see from Eq. (3) that the polynomial S is a function of the creation operators a_j^+ $(j = 1, 2, 3, 4, 5)$ and independent of the annihilation operators. Therefore, we can obtain the formula

$$[H_1, S] = a_1^+ [a_2 a_3 a_4 a_5, S] + a_5^+ a_4^+ a_3^+ a_2^+ [a_1, S]$$

* Supported by the National Key Basic Research and Development Programme of China under Grant No 2001CB309310, and the National Natural Science Foundation of China under Grant Nos 90103026 and 10125419.

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$$= a_1^+ \frac{\partial^4 S}{\partial a_2^+ \partial a_3^+ \partial a_4^+ \partial a_5^+} + a_5^+ a_4^+ a_3^+ a_2^+ \frac{\partial S}{\partial a_1^+}.$$

The polynomial S in Eq. (3) satisfies the following operator-type differential equation:

$$\Big(x_1 \frac{\partial^4}{\partial x_2 \partial x_3 \partial x_4 \partial x_5} + x_2 x_3 x_4 x_5 \frac{\partial}{\partial x_1}\Big) S = \lambda S. \quad (4)$$

Then it is straightforward to obtain the iterating equation from Eqs. (3) and (4) for the coefficients α_n of the polynomial S as follows:

$$\alpha_{j+1} = \lambda \alpha_j - p_{j-1} \alpha_{j-1}, \quad (5)$$

where

$$0 \le j \le q,\ q = \min(M, N, K, L),\ a_{-1} = a_{q+1} \equiv 0,$$
$$p_j = (j+1)(M-j)(N-j)(K-j)(L-j).$$

We will give the solution to Eq. (5) and the proof later.[14–19] The solution is

$$\alpha_j = \alpha_0 \det A^{(j)}(\lambda), \quad j = 1, 2, \ldots, q, \quad (6)$$

with the parameter λ determined by

$$\det A^{(q+1)}(\lambda) = 0,\ q = \min(M, N, K, L), \quad (7)$$

where $\det A^{(0)}(\lambda) \equiv 1$ and the matrices $A^{(k)}(\lambda)$ ($k \ge 1$) are $k \times k$ trigonal matrices with the matrix elements $A_{ij}^k(\lambda) = \lambda \delta_{i,j} - \delta_{i+1,j} - p_j \delta_{i,j+1}$ ($i, j = 0, 1, \ldots k-1$). Expanding Eq. (7) yields the explicit form of $\det A^{(n)}$,

$$\det A^{(n)} = \lambda^n + \sum_{k=1}^{\mathrm{int}(n/2)} (-1)^k \lambda^{n-2k} \Big(\sum_{i_1=2(k-1)}^{n-2} \cdot \sum_{i_2=2(k-2)}^{i_1-2} \cdots \sum_{i_j=2(k-j)}^{i_{j-1}-2} \cdots \cdot \sum_{i_k=0}^{i_{k-1}-2} p_{i_1} p_{i_2} \cdots p_{i_j} \cdots p_{i_k} \Big), \quad n \ge 2, \quad (8)$$

where $\mathrm{int}(n/2)$ represents the integer part of $n/2$, $\det A^{(0)}(\lambda) \equiv 1$, $\det A^{(1)}(\lambda) = \lambda$, and $p_j = (j+1)(M-j)(N-j)(K-j)(L-j)$.

The following is the brief proof of Eqs. (6) and (7). Indeed, Eq. (5) can be expressed as $A^{(q+1)}(\lambda)\alpha = 0$, $\alpha = (\alpha_0, \alpha_2, \ldots, \alpha_q)^T$. This results evidently in Eq. (7). Then we prove Eq. (6). Expanding the $(n+1) \times (n+1)$ trigonal determinant $\det A^{(n+1)}(\lambda)$ by the $(n+1)$th column leads to $\det A^{(n+1)} = \lambda \det A^{(n)} + D$, where D is an $n \times n$ determinant obtained by deleting both the nth row and $(n+1)$th column of the determinant $\det A^{(n+1)}$. Obviously, only one nonzero element $-p_{n-1}$ exists in the last row of the determinant D and hence $D = -p_{n-1} \det A^{(n-1)}$. The final result is $\det A^{(n+1)} = \lambda \det A^{(n)} - p_{n-1} \det A^{(n-1)}$, which has the same form as the one in Eq. (5) for α_n/α_0. Therefore, $\alpha_0 \det A^{(n)}$ is the solution to Eq. (5).

Consequently, using Fock states $|n, m, k, l, s\rangle = (\sqrt{n!m!k!l!s!})^{-1} {a_1^+}^n {a_2^+}^m {a_3^+}^k {a_4^+}^l {a_5^+}^s |\mathrm{vac}\rangle$, we obtain the resulting eigenenergies $E \equiv E_{M,N,K,L;\lambda}$ and eigenstates $|\psi\rangle \equiv |\psi_{M,N,K,L;\lambda}\rangle$ in terms of a parameter λ:

$$E = M\omega_2 + N\omega_3 + K\omega_4 + L\omega_5 + g\lambda, \quad (9a)$$

$$|\Psi\rangle = \sum_{j=0}^{q} c_n |j, M-j, N-j, K-j, L-j\rangle,$$

$$c_n = c_0 \det A^{(j)}(\lambda) \sqrt{\frac{(M-j)!(N-j)!(K-j)!(L-j)!}{j!M!N!K!L!}}, \quad (9b)$$

where $c_0 = \sqrt{M!N!K!L!}$; α_0 is a normalization constant; $q = \min(M, N, K, L)$, $M, N, K, L = 0, 1, 2 \cdots$; $\det A^{(n)}(\lambda)$ are given by Eq. (8), and the parameter λ is determined by Eq. (7). The proof is finished.

It is emphasized that Eq. (8) gives explicitly the analytical expressions of all energy eigenvalues and eigenstates utilizing the parameter λ for the models (1a). It is also easy to determine λ from Eq. (7). Comparing Eq. (7) with Eq. (8), we can conclude that the reduced equations of λ are in the form $W_k(\lambda^2) = 0 (\lambda G_k(\lambda^2) = 0)$ for an odd (even) integer q with $k = \mathrm{int}((q+1)/2)$. $W_k(x)$ and $G_k(x)$ are the polynomials of order k for variable x. The last step to obtain λ is to solve $G_k(x) = 0$ with $\lambda = 0, \pm\sqrt{x}$ for an even integer q or solve $W_k(x) = 0$ with $\lambda = \pm\sqrt{x}$ for an odd integer q. This implies that we only need solve a polynomial of order $\mathrm{int}((q+1)/2)$. Using λ and Eq. (8) it is easy to show the explicit analytical expressions of all the energy eigenvalues and eigenstates for model (1a).

It is clear that $\lambda = 0$ is a solution to Eq. (7) if $q = \min(M, N, K, L) \ge 0$ is an even integer. With the above analysis we can obtain eigenenergies $E \equiv E_{M,N,K,L;\lambda}$ and eigenstates $|\psi\rangle \equiv |\psi_{M,N,K,L;\lambda}\rangle$ under the condition $\lambda = 0$, i.e.

$$E = M\omega_2 + N\omega_3 + K\omega_4 + L\omega_5, \quad (10a)$$

$$|\Psi\rangle = \sum_{l=0}^{q/2} \rho_n |2l, M-2l, N-2l, K-2l, L-2l\rangle,$$

$$\rho_n = C \frac{\sqrt{(2l)!(M-2l)!(N-2l)!(K-2l)!(L-2l)!}}{(2l)!!(M-2l)!!(N-2l)!!(K-2l)!!(L-2l)!!}, \quad (10b)$$

where $M, N, K, L = q, q+1, \ldots,$ with $q = \min(M, N, K, L) \ge 0$ being non-negative integers, C is a normalization constant, $(2m)!! = 2m(2m-2)\cdots 2$ and $(2m+1)!! = (2m+1)(2m-1)\cdots 1$. We can find that the energy eigenvalues and eigenstates have especial simple forms, which demonstrate the situation without any interactions among the different modes.

In the case of $\lambda \neq 0$, using a several-line Mathematica code we can obtain explicit analytical expressions for $q = \min(M, N, K, L) \leq 8$ in a few minutes on a PC. We only list the results for $q \leq 3$ (there are $(q+1)$ values of λ for a fixed q),

$$q = 0 \Rightarrow \lambda = 0, \tag{11a}$$

$$q = 1 \Rightarrow \lambda = \pm\sqrt{QRS}, \tag{11b}$$

$$q = 2 \Rightarrow \lambda = 0, \pm\sqrt{2[(Q-1)(R-1)(S-1)+QRS]}, \tag{11c}$$

$$q = 3 \Rightarrow \lambda = \pm\sqrt{A \pm \sqrt{A^2 - 9QRS(Q-2)(R-2)(S-2)}}, \tag{11d}$$

$$A = (Q-5)(R-5)(S-5) - 17(R+Q+S) + 111,$$

where Q, R, and S are the three largest numbers in M, N, K, and L. The results imply that the simple and explicit forms of the eigenenergies are decided by nonnegative integers.

We can give the results for model (1b) without detail derivation. The eigenenergies and eigenstates corresponding to model (1b) are

$$E = M\omega_2 + N\omega_3 + K\omega_4 + L\omega_5 + g\lambda, \tag{12a}$$

$$|\psi_{M,N,K,L;\lambda}\rangle = S(a_1^+, a_2^+, a_3^+, a_4^+, a_5^+)|\text{vac}\rangle, \tag{12b}$$

where

$$S(x_1, x_2, x_3, x_4, x_5) = \sum_{j=0}^{q} \frac{\alpha_j}{j!} x_1^j x_2^{M-j} x_3^{N+j} x_4^{K-j} x_5^{L-j}$$

satisfies the differential equation

$$\left(x_1 x_3 \frac{\partial^3}{\partial x_2 \partial x_4 \partial x_5} + x_2 x_4 x_5 \frac{\partial^2 S}{\partial x_1 \partial x_3}\right) S = \lambda S, \tag{13}$$

$M, N, K, L = 0, 1, 2 \ldots$ are the nonnegative integers and the eigenvalues of the operators $M = a_1^+ a_1 + a_2^+ a_2$, $N = -a_1^+ a_1 + a_3^+ a_3$, $K = a_1^+ a_1 + a_4^+ a_4$, and $L = a_1^+ a_1 + a_5^+ a_5$, respectively; $x_j = a_j^+ (j = 1, 2, 3, 4, 5)$ and $q = \min(M, K, L)$. We have taken $n_3 \geq n_1$ (n_j denotes the occupying number of mode j) because the model (1b) is symmetry between the modes corresponding to $a_1^+ a_1$ and $a_3^+ a_3$. With $p_j = (j+1)(M-j)(N+j+1)(K-j)(L-j)$ Eqs. (5)–(10) still have the same form except Eq. (9b) and Eq. (10b) with the replacement $N - j \to N + j$. In summary, we have demonstrated the explicit analytical expressions of all the energy eigenvalues and eigenstates in terms of a parameter λ without using the assumption of Bethe ansatz for two different kinds of five-bosonic-mode mixing models as expressed in Eq. (1). The parameter λ can be numerated either analytically or numerically with a polynomial of order $\text{int}((q+1)/2)$. An explicit expression of the polynomial has been obtained. It is obvious that the procedure to determine the parameter λ is much simpler than solving a very complicated reduced system of Bethe equations for the unknown Bethe parameters in previous studies. When we obtain λ, it is fairly easy to obtain all the energies and eigenstates without any unknown parameter.

We have obtained the explicit analytical results of all the energy eigenvalues and eigenstates both for $q = \min(M, N, K, L) \leq 6$ and for the case of arbitrary nonnegative q with $\lambda = 0$. For $q \geq 7$, the parameter λ can be determined numerically. The analytical results of all energy eigenvalues and eigenstates are immediately obtained without solving the complex equations for the Bethe ansatz parameters.

The authors would like to thank Dr. Ying Wu for stimulating discussion.

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