A Scheme for Dense Coding in the Non-Symmetric Quantum Channel *

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We investigate the dense coding in the case of non-symmetric Hilbert spaces of the sender and receiver's particles sharing the quantum maximally entangled state. The efficiency of gaining classical information is also considered. We conclude that when a more level particle is with the sender, she can obtain a non-symmetric quantum channel from a symmetric one by entanglement transfer. Thus the efficiency of information transmission is improved.

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The quantum entanglement state [1−6] is an important tool distinguishing the quantum mechanics from the classical physics. Due to the non-local properties of the quantum entanglement state, it has been widely used in many fields such as quantum teleportation,[7−13] quantum dense coding [14−17] and quantum key distribution.[18−20] Here we focus our attention on quantum dense coding. In 1992, Bennett and Wiesner proposed the first scheme for the quantum dense coding,[14] in which Bell states are used as quantum channel. In 1996, quantum dense coding was experimentally presented by Mattle *et al.* in an optical system.[15] Recently Liu *et al.* presented a protocol for dense coding with multi-level entangled states.[16] However, in these schemes the quantum channels are symmetric, that is to say, the dimension of the Hilbert space of the particle with sender is the same as that of the particle with receiver. In 1998, Bose *et al.* generalized Bennett and Wiesner's scheme for dense coding to multi-parties: one receiver intends to receive messages from N senders.[17] In this case, the quantum channel is non-symmetric. The particles with senders are in 2^N-dimensional Hilbert space, while the particle with two-dimensional Hilbert space belongs to the receiver. In this Letter, we give a scheme for quantum dense coding in the case of non-symmetric Hilbert spaces of the sender and receiver's particles sharing the quantum maximally entangled state, and consider the efficiency of gaining classical information.

In Bennett and Wiesner's first scheme for the quantum dense coding, sender Alice and receiver Bob share a pair of entangled particles in the Bell state. Alice performs one of the four 1-qubit unitary operations given by the identity I or the Pauli matrices $(\sigma_x, i\sigma_y, \sigma_z)$ on her particle. Each of the unitary operations maps the Bell state to a different member of the four Bell states. Then Alice sends her particle to Bob. Bob can obtain two bits of classical information from the joint measurement on two particles. Now we wish to generalize Bennett and Wiesner's symmetric scheme to the non-symmetric one.

To present our scheme clearly, let us first begin with dense coding between two parties in 3×2 dimensions. Suppose the particle 1 in three-dimensional Hilbert space belongs to Alice, and particle 2 in two-dimensional Hilbert space is with Bob. They share the maximally entangled state:

$$|\Psi_{00}\rangle_{12} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12}. \tag{1}$$

Through simple calculation, it can be shown that the single-body operators on particle 1:

$$U_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U_{01} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$U_{10} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad U_{11} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$
$$U_{20} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad U_{21} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{2}$$

will transform $|\Psi_{00}\rangle$ into the corresponding state respectively,

$$U_{00}|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Psi_{00}\rangle, \tag{3}$$
$$U_{01}|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Psi_{01}\rangle, \tag{4}$$
$$U_{10}|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |21\rangle) = |\Psi_{10}\rangle, \tag{5}$$
$$U_{11}|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |21\rangle) = |\Psi_{11}\rangle, \tag{6}$$
$$U_{20}|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|20\rangle + |01\rangle) = |\Psi_{20}\rangle, \tag{7}$$
$$U_{21}|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|20\rangle - |01\rangle) = |\Psi_{21}\rangle. \tag{8}$$

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Alice operates one of the above unitary transformations, and sends her particle to Bob. Bob takes only one measurement in the basis $\{|\Psi_{00}\rangle, |\Psi_{01}\rangle, \ldots, |\Psi_{21}\rangle\}$, and he will know what operation Alice has carried out, that is, what messages Alice has encoded in the quantum state. As a result, Bob obtains $\log_2 6$ bits of information through only one measurement. Thus the dense coding is realized.

It is straightforward to generalize the above protocol to arbitrarily different dimensions for two parties. Assume that the Hilbert space of Alice's particle 1 is non-symmetric with that of Bob's particle 2, i.e. the dimension p of the Hilbert space of particle 1 is not the same as the dimension q of particle 2. Without loss of generality, we choose $p > q$. The non-symmetric quantum channel shared by Alice and Bob is the maximally entangled state

$$|\Psi_{00}\rangle_{12} = \frac{1}{\sqrt{q}}(|00\rangle + |11\rangle + \cdots + |q-1\,q-1\rangle)_{12}. \quad (9)$$

Clearly, an orthogonal basis can be composed of the general Bell states of the particles 1 and 2,

$$|\Psi_{mn}\rangle = \sum_j \mathrm{e}^{2\pi i j n/q}|(j \oplus m)\mathrm{mod}p\rangle_1 \otimes |j\rangle_2/\sqrt{q}, \quad (10)$$

where $m = 0, 1, \ldots, p-1$; $n, j = 0, 1, \ldots, q-1$. The single-body operators on particle 1 satisfying $U_{mn}|\Psi_{00}\rangle = |\Psi_{mn}\rangle$ may be explicitly written as

$$U_{mn} = \sum_j \mathrm{e}^{2\pi i j n/q}|(j \oplus m)\mathrm{mod}p\rangle\langle j|, \quad (11)$$

where $m = 0, 1, \ldots, p-1$; $n, j = 0, 1, \ldots, q-1$. The procedure for realizing dense coding in this case is similar to the above one. In order to send information, Alice performs one of the operations in Eq. (11) on her particle and sends it to Bob. Bob then performs a collective measurement in the general Bell basis $\{|\Psi_{mn}\rangle_{12} | m = 0, 1, \ldots, p-1;\ n = 0, 1, \ldots, q-1\}$ to find out what Alice has carried out on particle 1 and therefore reads out the encoded message. In this case, Bob can gain $\log_2(p \times q)$ bits of information. If $p = q$, the quantum channel becomes symmetric, correspondingly the bits of information become $\log_2 p^2 = 2\log_2 p$.

When the quantum channel between Alice and Bob is symmetric, as well as a more level particle being with Alice, the quantum channel may be transformed into a non-symmetric one, and hence improve the efficiency of the dense coding. For clarity, we start our discussion with the following simple case.

Suppose that the initial quantum channel between Alice and Bob is

$$|\Phi\rangle_{12} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12}, \quad (12)$$

where particle 1 belongs to Alice, and particle 2 to Bob, and they are both in two-dimensional Hilbert space. Besides the particle 3 belonging to Alice being in three-dimensional Hilbert space, its initial state is $|0\rangle_3$. Therefore, the initial state of the three particles is

$$|\Psi\rangle_{123} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12}|0\rangle_3. \quad (13)$$

Next we will swap the entanglement between particles 1 and 2 to the entanglement between particles 2 and 3 by using two unitary operations U_{13} and U'_{13} on particles 1 and 3. In other words, we want to transfer the state $|\Psi\rangle_{123}$ to the state $|0\rangle_1 \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{23}$ by the two unitary operations. In order to complete our task, we may design U_{13} to make $U_{13}|\Psi\rangle_{123} = \frac{1}{\sqrt{2}}\sum_{i=0}^{1}|iii\rangle_{123}$, the maximally entangled state, then U'_{13} should be constructed so that $U'_{13}\frac{1}{\sqrt{2}}\sum_{i=0}^{1}|iii\rangle_{123} = |0\rangle_1\frac{1}{\sqrt{2}}(|00\rangle_{23} + |11\rangle_{23})$. It is not difficult to prove that if Alice applies the following unitary operation on her particles 1 and 3 under the basis $\{|00\rangle_{13}, |01\rangle_{13}, |02\rangle_{13}, |10\rangle_{13}, |11\rangle_{13}, |12\rangle_{13}\}$:

$$U_{13}(2,3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (14)$$

where 2 and 3 in $(2, 3)$ of above the equation represent the dimensions of the Hilbert spaces of particles 1 and 3 respectively, then correspondingly $|\Psi\rangle_{123}$ becomes

$$U_{13}(2,3)|\Psi\rangle_{123} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{123}. \quad (15)$$

After that, Alice chooses and applies the following unitary operation on her particles 1 and 3:

$$U'_{13}(2,3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (16)$$

and hence Eq. (15) becomes

$$\begin{aligned} U'_{13}(2,3)\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{123} &= |0\rangle_1\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{23} \\ &= |0\rangle_1|\Phi'\rangle_{23}. \end{aligned} \quad (17)$$

As a result, the entanglement between particles 1 and 2 is transferred to the entanglement between particles 2 and 3. Before this, using the quantum channel between particles 1 and 2 for dense coding, Bob obtains $\log_2 4$ bits of information, but after that, Bob can gain $\log_2 6$ bits of information, so the efficiency of information transmission is increased.

In the following, we consider the general case. Assume that Alice and Bob share the maximally entangled state

$$|\Phi_{00}\rangle_{12} = \frac{1}{\sqrt{q}}(|00\rangle + |11\rangle + \cdots + |q-1\,q-1\rangle)_{12}, \quad (18)$$

where particle 1 is with Alice, particle 2 is with Bob, and they are both in q-dimensional Hilbert space. Besides particle 3 belonging to Alice being in p-dimensional Hilbert space, its initial state is $|0\rangle_3$, and $p > q$. Therefore, three particles will be in the state $|\Phi_{00}\rangle_{12} \otimes |0\rangle_3$. By generalizing $U_{13}(2,3)$ in Eq. (14) and $U'_{13}(2,3)$ in Eq. (16) to this general case, Alice may choose and perform the following unitary operations on her particles 1 and 3:

$$U_{13}(q,p) = \sum_{i=0}^{q-1} |ii\rangle\langle i0| + \sum_{j=1}^{q-1} |j0\rangle\langle jj| + \sum_{m,n} |mn\rangle\langle mn|, \quad (19)$$

where $m = 0, 1, \ldots, q-1$; $n = 1, 2, \ldots, p-1$, and $m \neq n$;

$$U'_{13}(q,p) = \sum_{i=0}^{q-1} |0i\rangle\langle ii| + \sum_{j=1}^{q-1} |jj\rangle\langle 0j| + \sum_{m,n} |mn\rangle\langle mn|, \quad (20)$$

in the sum of the above equation, when $m = 0$, we let $n = q, q+1, \ldots, p-1$; when $m = 1, 2, \ldots, q-1$, then n should take $n = 0, 1, \ldots, p-1$ with the condition $m \neq n$. After that $|\Phi_{00}\rangle_{12} \otimes |0\rangle_3$ correspondingly becomes

$$U'_{13}(q,p)U_{13}(q,p)|\Phi_{00}\rangle_{12} \otimes |0\rangle_3 = |0\rangle_1 \otimes \frac{1}{\sqrt{q}}(|00\rangle + |11\rangle + \cdots + |q-1\,q-1\rangle)_{23}. \quad (21)$$

Therefore, the entanglement transfer between particles 1 and 2 and particles 2 and 3 has been successfully realized. In other words, a symmetric quantum channel has been transformed into the non-symmetric one. Clearly, before this, Bob obtains $\log_2 q^2$ bits of information; now Bob can gain $\log_2 pq$ bits of information, and hence the efficiency of information transmission can be increased.

In summary, we have given a scheme to transform the symmetric level quantum maximally entangled state between two particles into a non-symmetric one, and used it as the quantum channel for dense coding. We have also considered the efficiency of information transmission. Comparing our scheme with the one for the symmetric dense coding, ours is obviously more efficient. We hope this scheme can be realized by experiments.

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