Quantum Inequality for Negative Energy Density States of Massive Dirac Field in Four-Dimensional Spacetime *

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Negative energy density and the quantum inequality are examined for the Dirac field. A proof is given of the quantum inequality for negative energy densities in the massive Dirac field produced by the superposition of two single particle electron states.

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Quantum inequalities constraining the magnitude of negative energy when it appears in quantum field theory and its duration have attracted much attention in recent years.[1−17] The reason is that the local energy density can be negative in quantum field theory and as a result all the known pointwise energy conditions in classical general relativity, such as the weak energy condition and null energy condition, are allowed to be violated. If no restrictions exist on negative energy densities, then they might be used to produce gross macroscopic effects such as violation of the second law of thermodynamics,[18,19] traversable wormholes,[20,21] "warp drive",[22] and even time machines.[21,23] Therefore, great effort has been devoted to determining the extent to which these violations of local energy are permitted in quantum field theory. In this respect, the Dirac field has been subject to much research recently. It has been demonstrated in Ref. [12] by Vollick that the energy density can be negative for a quantum state in the Dirac field, which is in the form of

$$|\psi\rangle = \frac{1}{\sqrt{1+\lambda^2}}(|k_z,1\rangle + \lambda|k_x,2\rangle), \tag{1}$$

where $|k_z,1\rangle$ and $|k_x,2\rangle$ are the single particle electron states and λ is real. Vollick[12] claims to have shown that this state, in the limit $m \to 0$, satisfies the quantum inequality which is obeyed by a massless scalar field.[2,3] Here we would like to point out that the proof of the quantum inequality given in Ref. [12] should only apply in the ultrarelativistic limit with $\omega_{k_x} \gg \omega_{k_z}$. This can be seen by noting that Eq. (23) in Ref. [12] is obtained in the above limits, and thus the sampled energy density,

$$\hat{\rho} = \frac{t_0}{\pi}\int_{-\infty}^{\infty}\frac{\langle\rho\rangle}{t^2+t_0^2}dt, \tag{2}$$

evaluated at $x = 0$ and in the limit $m \to 0$, is not given by that of Eq. (26) in Ref. [12], but by

$$\hat{\rho} = \frac{1}{(1+\lambda^2)V}\left[\omega_z + \lambda^2\omega_x + \frac{\lambda}{2}(\omega_x+\omega_z)e^{-|\omega_x-\omega_z|t_0}\right]. \tag{3}$$

Note that $\omega_x \neq \omega_z$. Therefore a question arises: whether the quantum state of Eq. (1) satisfies the quantum inequality in the limit $m \to 0$ as claimed in Ref. [12], or whether it just obeys the quantum inequality in the ultrarelativistic limit with $\omega_{k_x} \gg \omega_{k_z}$ as has been actually shown there. In this Letter, we address this question and show that the quantum inequality does hold in the limit $m \to 0$. Furthermore, we provide a proof of the quantum inequality without imposing $m \to 0$. Our plan is first to prove the inequality in the limit $m \to 0$ and then to give an argument which ensures that the inequality is automatically satisfied for $m \neq 0$ if it is for $m \to 0$.

Let $\omega_z = 2\pi n_z/L$ and $\omega_x = \pi n_x/L$, then it follows that

$$\begin{aligned}\hat{\rho} &= \frac{2\pi}{(1+\lambda^2)VL}\Big[n_z + \lambda^2 n_x \\ &\quad + \frac{\lambda}{2}(n_x+n_z)e^{-(2\pi/L)|n_x-n_z|t_0}\Big] \\ &\geq \frac{2\pi}{VL}\left[n_z + \lambda^2 n_x + \frac{\lambda}{2}(n_x+n_z)e^{-(2\pi/L)|n_x-n_z|t_0}\right].\end{aligned} \tag{4}$$

Making use of the results proven by Vollick,[12] we can see that the quantum inequality is satisfied if

$$\hat{\rho} \geq -\frac{2\sqrt{3}\pi e^{-\alpha}(1+e^{-\alpha})^2}{LV(1-e^{-\alpha})^4}, \tag{5}$$

or if

$$\begin{aligned}&-\left[n_z + \lambda^2 n_x + \frac{\lambda}{2}(n_x+n_z)e^{-\frac{\alpha}{2}|n_x-n_z|}\right] \\ &\leq \frac{\sqrt{3}e^{-\alpha}(1+e^{-\alpha})^2}{(1-e^{-\alpha})^4}.\end{aligned} \tag{6}$$

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Here $\alpha = 4\pi t_0/L$. For given n_x and n_z, the maximum value of the left hand is

$$f_{\max} = -n_z + \frac{n_x^2 + 2n_x n_z + n_z^2}{16n_x} e^{-\alpha|n_x - n_z|} < -\frac{7}{8}n_z + \frac{n_x^2 + n_z^2}{16n_x} e^{-\alpha|n_x - n_z|}, \quad (7)$$

where we have used the fact that $n_z e^{-\alpha|n_x - n_z|}/8 < n_z/8$. In order to show that Eq. (6) holds, we will deal with the case in which $n_x > n_z$ and that in which $n_z > n_x$ separately. First, if $n_x > n_z$, we have

$$f_{\max} < -\frac{7}{8}n_z + \frac{n_x^2 + n_z^2}{16n_x} e^{-\alpha(n_x - n_z)} < \Big(-\frac{7}{8}n_z + \frac{n_x^2 + n_z^2}{16n_x}\Big) e^{-\alpha(n_x - n_z)} < \frac{(n_x - n_z)}{16} e^{-\alpha(n_x - n_z)}. \quad (8)$$

It is easy to show that the right hand side is maximized for $n_x - n_z = 1/\alpha$. Thus

$$f_{\max} < \frac{1}{16\alpha} e^{-1}. \quad (9)$$

Now, following Vollick,[12] one can show that

$$\frac{1}{16\alpha} e^{-1} \le \frac{\sqrt{3} e^{-\alpha}(1 + e^{-\alpha})^2}{(1 - e^{-\alpha})^4}. \quad (10)$$

Therefore Eq. (6) holds and the quantum inequality is satisfied. Now let us turn our attention to the $n_z > n_x$ case. Here we have

$$\hat{\rho} = \frac{2\pi}{(1+\lambda^2)VL}\Big[n_z + \lambda^2 n_x + \frac{\lambda}{2}(n_x + n_z) e^{-\frac{2\pi}{L}|n_x - n_z|t_0}\Big] \ge \frac{2\pi}{\lambda^2 VL}\Big[n_z + \lambda^2 n_x + \frac{\lambda}{2}(n_x + n_z) e^{-\frac{2\pi}{L}|n_x - n_z|t_0}\Big] = \frac{2\pi}{VL}\Big[n_x + \frac{1}{\lambda^2} n_z + \frac{1}{2\lambda}(n_x + n_z) e^{-\frac{2\pi}{L}|n_z - n_x|t_0}\Big]. \quad (11)$$

The above expression becomes exactly Eq. (4) after introducing the following transformation:

$$\lambda \to \frac{1}{\lambda}, \quad n_x \to n_z. \quad (12)$$

Therefore, our previous proof for the $n_x > n_z$ case applies. This completes our proof for the inequality for $m \to 0$. Now we show that the inequality holds automatically for $m \neq 0$. To see this, let us note that, for $m \neq 0$, the sampled energy density at $x = 0$ is given by

$$\hat{\rho} = \frac{1}{(1+\lambda^2)V}[\omega_z + \lambda^2\omega_x + \lambda a e^{-|\omega_x - \omega_z|t_0}], \quad (13)$$

where

$$a = \frac{k_x k_z(\omega_x + \omega_z)}{2\sqrt{\omega_x\omega_z(\omega_x + m)(\omega_z + m)}} \le \frac{k_x k_z(\omega_x + \omega_z)}{2\omega_x\omega_z} \le \frac{1}{2}(\omega_x + \omega_z). \quad (14)$$

Therefore we have

$$\hat{\rho} = \frac{1}{(1+\lambda^2)V}[\omega_z + \lambda^2\omega_x + \lambda a e^{-|\omega_x - \omega_z|t_0}] \ge \frac{1}{(1+\lambda^2)V}\Big[\omega_z + \lambda^2\omega_x + \frac{\lambda}{2}(\omega_x + \omega_z) e^{-|\omega_x - \omega_z|t_0}\Big]. \quad (15)$$

A comparison of the above expression with Eq. (3) reveals that the quantum inequality is satisfied for $m \neq 0$ if it is for $m \to 0$. It is noted that in the above derivation, we have assumed $\lambda < 0$, since if otherwise, $\hat{\rho}$ is positive and thus Eq. (5) is automatically satisfied.

In summary, we have given a proof of the claim made in Ref. [12], i.e. the quantum state that is the superposition of two single particle states satisfies, in the limit $m \to 0$, the quantum inequality for massless scalar fields, and furthermore, we have shown that the restriction of $m \to 0$ can be removed.

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