An Exact Solution to the Master Equation for the Dissipative Harmonic Oscillator Driven by an External Field *

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By using the algebraic structure in the operator dual space in the master equation for the driven dissipative harmonic oscillator, we have rewritten the master equation as a Schrödinger-like equation. Then we have used three gauge transformations and obtained an exact solution to the master equation in the particle number representation.

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All physical systems cannot be completely isolated from their environment, their processes are associated with a loss mechanism, so the quantum theory of damping is an important aspect in quantum mechanics or quantum optics.[1−3] The damping of a system is described by its interaction with a reservoir, the reservoir can be described by a large number of harmonic oscillators as the modes of the free electromagnetic field or phonon modes in a solid. The density operator approach is a main means to develop the quantum theory of damping. In general, we are interested in the statistical properties of the system and not in the details of the reservoir, then the reservoir degree of freedom is traced out and the reduced density operator for the system is obtained. The usual way to solve master equations of the reduced density is to convert a master equation to a c-number equation by using the Glauber–Sudarshan representation.[1−3]

The dissipative harmonic oscillator driven by an external field represents losses in a cavity pumped by an external field or dissipation by phonons in a solid.[1,4−6] Some authors applied the norm-ordering techniques in the coherent state representation to discuss the solution of the master equation.[4−6] In this Letter, we use the algebraic dynamical method[7,8] to solve the master equation. The first step of the algebraic method is to construct the right and left algebras together with their combined algebras, then the master equation for the reduced density is written as a Schrödinger-like equation. The Liouvillian superoperator behaving as the Hamiltonian in the Schrödinger equation is written as a linear combination of the generators of the above algebras. The next step is to use the gauge transformations to change the Liouvillian superoperator into a function of the Cartan operators of the algebras. The exact solutions can be obtained by using the inverse transformation. This method has been used to solve the von Neumann equation for the quantum statistical characteristic function of the two-level Jaynes–Cummings model and the master equation for the sympathetic cooling of the Bose–Einstein condensate system in the mean field approximation.[9,10]

The Hamiltonian for the dissipative harmonic oscillator driven by an external field in the rotating wave approximation reads[1,2,4−6]

$$\hat{H} = \hat{H}_0 + \hat{H}_D + \hat{H}_I, \tag{1}$$

where

$$\hat{H}_0 = \omega \hat{a}^+ \hat{a} + \sum_i \omega_i \hat{b}_i^+ \hat{b}_i, \tag{1a}$$

$$\hat{H}_D = f(t) \mathrm{e}^{-\mathrm{i}\omega_0 t} \hat{a}^+ + f(t) \mathrm{e}^{\mathrm{i}\omega_0 t} \hat{a}, \tag{1b}$$

and

$$\hat{H}_I = \sum_k (g_k \hat{b}_k \hat{a}^+ + g_k^* \hat{b}_k^+ \hat{a}). \tag{1c}$$

$\hat{H}_0$ in Eq. (1a) is the free Hamiltonians for the oscillator and the reservoir, $\hat{H}_D$ in Eq. (1b) represents the term driven by an external field, the interaction between the system and the reservoir is described by the Hamiltonian $\hat{H}_I$ in Eq. (1c) where $\hat{a}$ (and $\hat{a}^+$) and $\hat{b}_i$ (and $\hat{b}_i^+$) are the destruction (and creation) operators of the system and the reservoir with frequencies ω and ω_i, respectively; g_k (and g_k^*) are the coupling constants between the system and the reservoir, $f(t)$ is the amplitude of the driven field with frequency ω_0. In the Markov approximation the master equation of the reduced density for the system in the interaction picture is[1−3]

$$\begin{aligned} \frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} =& -\mathrm{i}F(t)(\hat{a}^+\hat{\rho} - \hat{\rho}\hat{a}^+) - \mathrm{i}F^*(t)(\hat{a}\hat{\rho} - \hat{\rho}\hat{a}) \\ &+ \gamma N_R(\hat{a}\hat{\rho}\hat{a}^+ + \hat{a}^+\hat{\rho}\hat{a} - \hat{a}^+\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^+\hat{a} - \hat{\rho}) \\ &+ \frac{\gamma}{2}(2\hat{a}\hat{\rho}\hat{a}^+ - \hat{a}^+\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^+\hat{a}), \end{aligned} \tag{2}$$

where $F(t) = f(t)\mathrm{e}^{-\mathrm{i}\omega_0 t}$, γ is the damping constant, and N_R is the mean number of the reservoir quanta. If

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the reservoir is in thermal equilibrium at temperature T, $N_R = (\mathrm{e}^{\omega/KT} - 1)^{-1}$.

We introduce the left and right algebras based on the location where the destruction (and creation) operators of the system denote the left and the right-hand side of the density operator. The left and right algebras of the algebra $h\omega(4)$ denoted by $h\omega(4)_l$ and $h\omega(4)_r$ are given by[[9,10]]

$$\begin{aligned} &h\omega(4)_l: \\ &[\hat{n}_l, \hat{a}_l^+] = \hat{a}_l^+, \quad [\hat{n}_l, \hat{a}_l] = -\hat{a}_l, \\ &[\hat{a}_l, \hat{a}_l^+] = 1; \end{aligned} \tag{3}$$

$$\begin{aligned} &h\omega(4)_r: \\ &[\hat{n}_r, \hat{a}_r^+] = -\hat{a}_r^+, \quad [\hat{n}_r, \hat{a}_r] = \hat{a}_r, \quad [\hat{a}_r, \hat{a}_r^+] = -1, \end{aligned} \tag{4}$$

where the operators in $h\omega(4)_l$ act from the left on the ket state $|\ \rangle$ and the operators in the $h\omega(4)_r$ from the right on the bra state $\langle\ |$. Here $h\omega(4)_l$ (and $h\omega(4)_r$) are isomorphic (anti-isomorphic) to $h\omega(4)$. Since the operators in the $h\omega(4)_l$ and $h\omega(4)_r$ algebras act in different spaces, they commute with each other as follows

$$[\hat{A}_l, \hat{B}_r] = 0, \quad \hat{A}, \hat{B} = \{\hat{n}, \hat{a}, \hat{a}^+\}. \tag{5}$$

The operators in $h\omega(4)_l$ and $h\omega(4)_r$ can be combined to form the generators of the $SU(1,1)$ algebra,

$$\hat{K}_0 = \frac{1}{2}(\hat{n}_l + \hat{n}_r), \quad \hat{K}_+ = \hat{a}_l^+ \hat{a}_r, \quad \hat{K}_- = \hat{a}_r^+ \hat{a}_l. \tag{6}$$

They satisfy the relations

$$[\hat{K}_0, \hat{K}_\pm] = \pm \hat{K}_\pm, \quad [\hat{K}_-, \hat{K}_+] = 2\hat{K}_0 \tag{7}$$

and act on the projectors $\hat{P}^m = |m\rangle\langle m|$ in the following forms

$$\begin{aligned} &\hat{K}_0 \hat{P}^m = \frac{1}{2}(2m+1)\hat{P}^m, \ \hat{K}_+ \hat{P}^m = (m+1)\hat{P}^{m+1}, \\ &\hat{K}_- \hat{P}^m = m\hat{P}^{m-1}. \end{aligned} \tag{8}$$

The operator $\hat{C}_0 = \hat{n}_l - \hat{n}_r$ is a radical. Then the master Eq. (2) can be written as a Schrödinger-like equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = \hat{\Gamma}\hat{\rho}, \tag{9}$$

where the Liouvillian superoperator $\hat{\Gamma}$ is

$$\begin{aligned} \hat{\Gamma} = &-\mathrm{i}(F(t)\hat{a}_l^+ + F^*(t)\hat{a}_l) + \mathrm{i}(F(t)\hat{a}_r^+ + F^*(t)\hat{a}_r) \\ &+ \gamma(N_R+1)\hat{K}_- + \gamma N_R \hat{K}_+ \\ &- 2\gamma\left(N_R + \frac{1}{2}\right)\hat{K}_0 + \frac{\gamma}{2}. \end{aligned} \tag{10}$$

In the following we use the algebraic dynamical method to solve the master Eq. (9). We use the operator

$$\hat{R}(t) = \exp(\alpha_+ \hat{K}_+) \exp(\alpha_- \hat{K}_-), \tag{11}$$

to proceed the first gauge transformation

$$\hat{\Gamma}^{(1)} = \hat{R}(t)^{-1}\hat{\Gamma}\hat{R}(t) - \hat{R}(t)^{-1}\frac{\partial}{\partial t}\hat{R}(t), \tag{12}$$

$$\hat{\rho}^{(1)} = \hat{R}(t)^{-1}\hat{\rho}. \tag{13}$$

When the parameters $\alpha_\pm$ satisfy

$$\begin{aligned} &\gamma(N_R+1)(1-\alpha_+\alpha_-)^2 + \gamma N_R \alpha_-^2 + \gamma\alpha_-(2N_R+1) \\ &\quad \cdot (1-\alpha_+\alpha_-) - \dot{\alpha}_- - \dot{\alpha}_+\alpha_-^2 = 0, \end{aligned} \tag{14}$$

$$\gamma N_R + \gamma(N_R+1)\alpha_+^2 - \gamma\alpha_+(2N_R+1) - \dot{\alpha}_+ = 0, \tag{15}$$

the transformed superoperator is

$$\begin{aligned} \hat{\Gamma}^{(1)} = &\mathrm{i}F(t)(\alpha_+ - 1)\hat{a}_l^+ - \mathrm{i}F^*(t)[\alpha_- + (1-\alpha_+\alpha_-)]\hat{a}_l \\ &+ \mathrm{i}F(t)[\alpha_- + (1-\alpha_+\alpha_-)]\hat{a}_r^+ \\ &- \mathrm{i}F^*(t)(\alpha_+ - 1)\hat{a}_r + \gamma[2(N_R+1)\alpha_+ \\ &- (2N_R+1)]\hat{K}_0 + \frac{\gamma}{2}. \end{aligned} \tag{16}$$

By using the operator

$$\hat{D}_r(t) = \mathrm{e}^{u_1 \hat{a}_r} \mathrm{e}^{u_2 \hat{a}_r^+}, \tag{17}$$

we proceed the second gauge transformation

$$\hat{\Gamma}^{(2)} = \hat{D}_r(t)^{-1}\hat{\Gamma}^{(1)}\hat{D}_r(t) - \hat{D}_r(t)^{-1}\frac{\partial}{\partial t}\hat{D}_r(t), \tag{18}$$

$$\hat{\rho}^{(2)} = \hat{D}_r(t)^{-1}\hat{\rho}^{(1)}. \tag{19}$$

The operator $\hat{\Gamma}^{(2)}$ in Eq. (18) is

$$\begin{aligned} \hat{\Gamma}^{(2)} = &\mathrm{i}F(t)(\alpha_+ - 1)\hat{a}_l^+ - \mathrm{i}F^*(t)[\alpha_- + (1-\alpha_+\alpha_-)]\hat{a}_l \\ &+ \gamma[2(N_R+1)\alpha_+ - (2N_R+1)]\hat{K}_0 \\ &+ \mathrm{i}F(t)u_1[\alpha_- + (1-\alpha_+\alpha_-)] + \frac{\gamma}{2} \end{aligned} \tag{20}$$

with the parameters u_1 and u_2 satisfying

$$\begin{aligned} &-\mathrm{i}F(t)[(1-\alpha_+\alpha_-) + \alpha_-] + \frac{\gamma u_2}{2}[2(N_R+1)\alpha_+ \\ &- (2N_R+1)] + \dot{u}_2 = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} &\mathrm{i}F^*(t)(\alpha_+ - 1) - \frac{\gamma u_1}{2}[2(N_R+1)\alpha_+ - (2N_R+1)] \\ &+ \dot{u}_1 = 0. \end{aligned} \tag{22}$$

Subsequently, by using operator

$$\hat{D}_l(t) = \mathrm{e}^{v_1 \hat{a}_l^+} \mathrm{e}^{v_2 \hat{a}_l}, \tag{23}$$

the last gauge transformation

$$\hat{\Gamma}^{(3)} = \hat{D}_l(t)^{-1}\hat{\Gamma}^{(2)}\hat{D}_l(t) - \hat{D}_l(t)^{-1}\frac{\partial}{\partial t}\hat{D}_l(t), \tag{24}$$

$$\hat{\rho}^{(3)} = \hat{D}_l(t)^{-1}\hat{\rho}^{(2)}, \tag{25}$$

leads to the results

$$\hat{\Gamma}^{(3)} = \gamma[2(N_R+1)\alpha_+ - (2N_R+1)]\hat{K}_0 + \mathrm{i}\eta(t), \tag{26}$$

where

$$\eta(t)=[\alpha_-+(1-\alpha_+\alpha_-)](F(t)u_1-F^*(t)v_1)-\mathrm{i}\frac{\gamma}{2}. \quad (27)$$

The parameters v_1 and v_2 in Eq. (26) obey the equations

$$-\mathrm{i}F^*(t)[\alpha_-+(1-\alpha_+\alpha_-)]-\frac{\gamma v_2}{2}[2(N_R+1)\alpha_+ -(2N_R+1)]-\dot{v}_2=0, \quad (28)$$

$$\mathrm{i}F(t)(\alpha_+-1)+\frac{\gamma v_1}{2}[2(N_R+1)\alpha_+ -(2N_R+1)]-\dot{v}_1=0. \quad (29)$$

The eigenvectors of the operator $\hat{\Gamma}^{(3)}$ in Eq. (26) are

$$\hat{P}^m=|m\rangle\langle m| \quad (30)$$

with the eigenvalues

$$\lambda_m=\frac{\gamma}{2}(2m+1)[2(N_R+1)\alpha_+-(2N_R+1)]+\mathrm{i}\eta(t).$$

The solution to the master equation for $\hat{\Gamma}^{(3)}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}^{(3)}=\hat{\Gamma}^{(3)}\hat{\rho}^{(3)} \quad (31)$$

is

$$\hat{\rho}^{(3)}(t)=\exp\Big[\int_0^t\lambda_m(t')\mathrm{d}t'\Big]\hat{P}^m. \quad (32)$$

By using an inverse transformation from Eqs. (13), (19), and (25), we can obtain the solution to the master equation for the total system

$$\hat{\rho}(t)=\exp\Big[\int_0^t\lambda_m(t')\mathrm{d}t'\Big]\hat{R}(t)\hat{D}_r(t)\hat{D}_l(t)\hat{P}^m. \quad (33)$$

Here we suppose that the initial values are $\alpha_+(0)=\alpha_-(0)=u_1(0)=u_2(0)=v_1(0)=v_2(0)=0$, then $\hat{R}(0)=\hat{D}_r(0)=\hat{D}_l(0)=1$ and $\hat{\rho}(0)=\hat{\rho}^{(3)}(0)$.

To obtain an explicit form of the solution to Eq. (33), we introduce the formulae

$$\exp(\alpha\hat{a}_r^+)\hat{a}_r^n=(\hat{a}_r+\alpha)^n\exp(\alpha\hat{a}_r^+),$$
$$\exp(\beta\hat{a}_l)(\hat{a}_l^+)^n=(\hat{a}_l^++\beta)^n\exp(\beta\hat{a}_l),$$
$$\exp(\gamma\hat{a}_l^+)(\hat{a}_l)^n=(\hat{a}_l-\gamma)^n\exp(\gamma\hat{a}_l^+).$$

Then we obtain

$$\hat{\rho}(t)=\mathrm{e}^{-\mathrm{i}\int_0^t\eta(t')\mathrm{d}t'}\mathrm{e}^{-\frac{(1-\alpha_+\alpha_-)u_1v_1}{\alpha_+}}\sum_{k,s=0}^{\infty}\sum_{l,p,q=0}^{m}\sum_{\mu,\nu=0}^{k}\cdot\Big\{\frac{(m!)^3k!(s+\nu)!\sqrt{(\mu+l)!}}{s!\mu!\nu!l!p!q!(m-s)!(k-\mu)!(k-\nu)!\sqrt{(s+\nu-p)!}}\cdot\mathrm{e}^{\frac{\gamma}{2}(2m+1)\int_0^t[2\alpha_+(N_R+1)-(2N_R+1)]\mathrm{d}t'}\cdot(1-\alpha_+\alpha_-)^l(\alpha_-v_1+u_2)^{m-s}\cdot v_2^qu_1^{k-\mu}v_1^{k-\nu}|\mu+l\rangle\langle s+\nu-p|\Big\},\ l+p+q=m. \quad (34)$$

In conclusion, we have constructed the right and left algebras together with their combined algebras for the driven dissipative harmonic oscillator. Based on the algebraic structure we have rewritten the master equation as a Schrödinger-like equation. Then we have obtained an exact solution to the master equation in the particle number representation by using the algebraic dynamical method.

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