Berry's Phase in Noncommutative Spaces *

S. A. Alavi**

High Energy Physics Division, Department of Physics, University of Helsinki, Finland
Helsinki Institute of Physics, FIN-00014 Helsinki, Finland

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We discuss the perturbative aspects of noncommutative quantum mechanics. Then we study Berry's phase within the framework of noncommutative quantum mechanics. The results show deviations from the usual quantum mechanics, which depend on the parameter of space/space noncommutativity.

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Noncommutative quantum mechanics have received a wide attention once it was realized that they could be obtained as low-energy limit of string theory in the presence of a B field.[1,2] In field theories the noncommutativity is introduced by replacing the standard product by the star product. For a manifold parametrized by the coordinates x_i, the noncommutative relation can be written as

$$[\hat{x}_i, \hat{x}_j] = \mathrm{i}\theta_{ij} \quad [\hat{x}_i, \hat{p}_i] = \mathrm{i}\delta_{ij} \quad [\hat{p}_i, \hat{p}_j] = 0. \tag{1}$$

Many physical problems have been studied within the framework of the noncommutative quantum mechanics (NCQM), as can be seen in Refs. [6–21]. The NCQM is formulated in the same way as the standard quantum mechanics (SQM) (quantum mechanics in commutative spaces), which is in terms of the same dynamical variables represented by operators in a Hilbert space and a state vector evolved according to the Schrödinger equation:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = H_{nc}|\psi\rangle, \tag{2}$$

we have taken into account $\hbar = 1$. $H_{nc} \equiv H_\theta$ denotes the Hamiltonian for a given system in the noncommutative space. In the literature, two approaches have been considered for constructing the NCQM:

(a) $H_\theta = H$, so that the only difference between SQM and NCQM is the presence of a nonzero θ in the commutator of the position operators, i.e., Eq. (1).

(b) By deriving the Hamiltonian from the moyal analogue of the standard Schrödinger equation:

$$\begin{aligned}\mathrm{i}\frac{\partial}{\partial t}\psi(x,t) &= H\Big(p = \frac{1}{i}\nabla, x\Big)\psi(x,t) \\ &\equiv H_\theta\psi(x,t),\end{aligned} \tag{3}$$

where $H(p,x)$ is the same Hamiltonian as that in the standard theory, and we observe the θ dependence enters into now through the star product.[19] In Ref. [21], it has been shown that these two approaches lead to the same physical theory. The Hamiltonian is in the form of

$$H(\hat{p},\hat{x}) = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{4}$$

The modified Hamiltonian H_θ can be obtained by a shift in the argument of the potential:[6,19]

$$x_i = \hat{x}_i + \frac{1}{2}\theta_{ij}\hat{p}_j \quad \hat{p}_i = p_i, \tag{5}$$

where $\theta_{ij} = \frac{1}{2}\epsilon_{ijk}\theta_k = \frac{1}{2}\epsilon_{ij}\theta$ (if $\theta_i = \theta, i = 1, 2, \cdots$). Then we have

$$H_\theta = \frac{p^2}{2m} + V\Big(x_i - \frac{1}{2}\theta_{ij}p_j\Big). \tag{6}$$

The variables x_i and p_i now satisfy the same commutation relations as in the usual case:

$$[x_i, x_j] = [p_i, p_j] = 0, \quad [x_i, p_j] = \delta_{ij}. \tag{7}$$

Topological effects are among the most important quantum phenomena discovered since the creation of the quantum mechanics. The Aharonov–Bohm (AB) effect[3] was the first such effect which changed basic assumptions about the role of the potentials in physics. There is also a geometric effect discovered by Berry[4] almost two decades later. Berry investigated quantum systems that undergo an adiabatic evolution governed by several changing parameters. When these parameters return to their initial values, quantum systems return to their initial states up to a phase. Berry found that the phase contains a geometric part, which depends on the path in the space of the parameters, but not on the rate evolution of the system along the path. Aharonov and Anandan subsequently generalized Berry's phase to nonadiabatic evolution.[5] The AB effect in noncommutative spaces has been studied in Refs. [7, 8]. In this Letter, we study Berry's phase in noncommutative spaces.

We discuss the perturbation aspects of noncommutative dynamics. The perturbation aspects of q-deformed dynamics in one-dimensional q-spaces has been studied in Refs. [22, 23]. Using

$$U(x + \Delta x) = U(x) + \sum_{n=1}^{\infty}\frac{U^{(n)}(x)}{n!}(\Delta x)^n \tag{8}$$

and Eq. (6), we have

$$H_{nc} = \frac{p^2}{2m} + V(x_i) + \sum_{n=1}^{\infty}\frac{V^{(n)}(x_i)}{n!}(\Delta x_i)^n, \tag{9}$$

* Supported partially by the Ministry of Science, Research and Technology of Iran.
** On leave of absence from: Department of Physics, Tarbiat Moallem University of Sabzevar, P.O. Box 397, Sabzevar, Iran
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where $\Delta x_i = -\frac{1}{2}\theta_{ij}p_j$ and $H = \frac{p^2}{2m} + V(x)$ is the Hamiltonian in ordinary (commutative) space. Since the noncommutativity parameter, if it is non-zero, should be very small compared with the length scales of the system, one can always treat the noncommutative effects as some perturbations of commutative counterpart and hence up to the first order in θ, we can write

$$\begin{aligned} H_{nc} &= \frac{p^2}{2m} + V(x_i) + \Delta x_i \frac{\partial V}{\partial x_i} \\ &= H + \Delta x_i \frac{\partial V}{\partial x_i} = H + \theta H_I. \end{aligned} \tag{10}$$

We can use perturbation theory to obtain the eigenvalues and eigenfunctions of H_{nc}:

$$E_n = E_n^0 + \Delta E_n^0 = E_n^0 + \theta E_n^{(1)} + \theta^2 E_n^{(2)} + \cdots, \tag{11}$$

$$\psi_n = \phi_n + \sum_{k\neq n} C_{nk}(\theta)\phi_k, \tag{12}$$

where

$$C_{nk}(\theta) = \theta C_{nk}^{(1)} + \theta^2 C_{nk}^{(2)} + \cdots. \tag{13}$$

To the first order in perturbation theory we have

$$\theta E_n^{(1)} = \langle\phi_n|\theta H_I|\phi_n\rangle, \tag{14}$$

$$\psi_n = \phi_n + \theta \sum_{k\neq n} C_{nk}^{(1)}\phi_k, \tag{15}$$

$$\theta C_{nk}^{(1)} = \frac{\langle\phi_k|\theta H_I|\phi_n\rangle}{E_n^0 - E_k^0}, \tag{16}$$

where E_n^0 and ϕ_n are the nth eigenvalue and eigenfunction of the Hamiltonian H. E_n and ψ_n are the nth eigenvalue and eigenfunction of H_{nc}. For example, for the hydrogen atom,

$$V(r) = -\frac{ze^2}{r} = -\frac{ze^2}{\sqrt{\hat{x}\hat{x}}}, \tag{17}$$

and we have

$$\theta H_I = -\frac{1}{2}\theta_{ij}p_j\left(\frac{x^i ze^2}{r^3}\right) = -ze^2\frac{x_i\theta_{ij}p_j}{2r^3}, \tag{18}$$

which is in agreement with the result of Ref. [6]. Using Eqs. (15) and (16) we can find the eigenfunctions for the hydrogen atom in noncommutative spaces,

$$\psi_n^{\text{hydrogen}} = \phi_n + \sum_{k\neq n}\frac{\langle\phi_k|\theta H_I|\phi_n\rangle}{E_n^0 - E_k^0}\phi_k, \tag{19}$$

where θH_I is given by Eq. (18).

Now the Hamiltonian H_{nc}, its eigenfunctions ψ_n and eigenvalues E_n are known (Eqs. (11)–(13)) and we can calculate Berry's phase in the noncommutative case. We suppose that the quantum system is in nth eigenstate, i.e.,

$$H_{nc}|\psi_n\rangle = E_n|\psi_n\rangle. \tag{20}$$

If H_{nc} is slowly changed, then according to the adiabatic theorem the system remains in the nth eigenstate of the slowly changing Hamiltonian H_{nc}. If it returns to its initial form $H_{nc}(\tau) = H_{nc}(t=0)$ at time τ, it follows that the system must return to its initial state up to a phase factor, and we have

$$|\psi_n(\tau)\rangle = \mathrm{e}^{-\mathrm{i}\int_0^\tau E_n(t')\mathrm{d}t'}\mathrm{e}^{\mathrm{i}\eta_n(\tau)}|\psi_n\rangle. \tag{21}$$

Now, the Schrödinger equation is

$$\mathrm{i}\frac{\partial|\psi_n(t)\rangle}{\partial t} = E_n(t)|\psi_n(t)\rangle. \tag{22}$$

Differentiating Eq. (21) at time t, and plugging into the Schrödinger equation, we obtain

$$\frac{\partial}{\partial t}|\psi_n(t)\rangle + \mathrm{i}\frac{\mathrm{d}\eta_n(t)}{\mathrm{d}t}|\psi_n(t)\rangle = 0. \tag{23}$$

Multiplying on the left by $\langle\psi_n(t)|$, we have

$$\eta_n(\tau) = \mathrm{i}\int_0^\tau \langle\psi_n(t)|\frac{\partial}{\partial t}|\psi_n(t)\rangle\mathrm{d}t. \tag{24}$$

If the time dependence of the Hamiltonian arises because we can change some parameters R with time, then we may write

$$\begin{aligned} \eta_n(\tau) &= \mathrm{i}\int_0^\tau \langle\psi_n(t)|\frac{\partial}{\partial t}|\psi_n(t)\rangle\mathrm{d}t \\ &= \mathrm{i}\oint \mathrm{d}R\langle\psi_n(R)|\nabla_R\psi_n(R)\rangle. \end{aligned} \tag{25}$$

By substituting for $|\psi_n(t)\rangle$ for a given time t:

$$|\psi_n(t)\rangle = |\phi_n(t)\rangle + \sum_{k\neq n} C_{nk}(\theta)|\phi_k(t)\rangle, \tag{26}$$

we obtain

$$\begin{aligned} \langle\psi_n(t)|\frac{\partial}{\partial t}|\psi_n(t)\rangle =& \Big[\langle\phi_n(t)| + \sum_{k\neq n} C_{nk}(\theta)\langle\phi_k(t)|\Big] \\ &\cdot \frac{\partial}{\partial t}\Big[|\phi_n(t)\rangle + \sum_{\ell\neq n} C_{n\ell}(\theta)|\phi_\ell(t)\rangle\Big], \end{aligned} \tag{27}$$

which leads to

$$\begin{aligned} \eta_n(\tau) =& \mathrm{i}\int_0^\tau \mathrm{d}t\langle\phi_n(t)|\frac{\partial}{\partial t}|\phi_n(t)\rangle \\ &+ \mathrm{i}\sum_{i=1}^\infty \theta^i \int_0^\tau \mathrm{d}t\Big\{\sum_{\ell\neq n} C_{n\ell}^{(i)}\langle\phi_n(t)|\frac{\partial}{\partial t}|\phi_\ell(t)\rangle \\ &+ \sum_{k\neq n} C_{nk}^{(i)}\langle\phi_k(t)|\frac{\partial}{\partial t}|\phi_n(t)\rangle \\ &+ \sum_{r=1}^\infty\sum_{\ell\neq n}\sum_{k\neq n}\theta^r C_{n\ell}^{(i)}C_{nk}^{(r)}\langle\phi_\ell(t)|\frac{\partial}{\partial t}|\phi_k(t)\rangle\Big\}, \end{aligned} \tag{28}$$

or

$$\begin{aligned} \eta_n =& \mathrm{i}\oint \mathrm{d}R\langle\phi_n(R)|\nabla_R\phi_n(R)\rangle \\ &+ \mathrm{i}\sum_{i=1}^\infty \theta^i \oint \mathrm{d}R\Big\{\sum_{\ell\neq n} C_{n\ell}^{(i)}\langle\phi_n(R)|\nabla_R\phi_\ell(R)\rangle \end{aligned}$$

$$+\sum_{k\neq n} C_{nk}^{(i)}\langle\phi_k(R)|\nabla_R\phi_n(R)\rangle$$
$$+\sum_{r=1}^{\infty}\sum_{\ell\neq n}\sum_{k\neq n}\theta^r C_{n\ell}^{(i)} C_{nk}^{(r)}\langle\phi_\ell(R)|\nabla_R\phi_k(R)\rangle\Big\}. \quad (29)$$

The first term is Berry's phase in commutative space and the second term gives the correction to Berry's phase due to the noncommutativity of space. If the parameter θ is small, the higher-order terms (higher powers in θ) will be very small and we can have

$$\eta_n = \mathrm{i}\oint \mathrm{d}R\langle\phi_n(R)|\nabla_R\phi_n(R)\rangle + \mathrm{i}\theta\oint \mathrm{d}R\Big\{\sum_{\ell\neq n} C_{n\ell}^{(1)}\langle\phi_n(R)|\nabla_R\phi_\ell(R)\rangle + \sum_{k\neq n} C_{nk}^{(1)}\langle\phi_k(R)|\nabla_R\phi_n(R)\rangle\Big\} + O(\theta^2). \quad (30)$$

As mentioned above, the argument can be generalized to nonadiabatic evolution. Next, we consider the same problem as in Ref. [24]. A particle of unit mass slides freely rounds a non-circular hoop, which is slowly turned through one complete rotation, in its own plane, about a centre. The rotation can be described by a single parameter X, namely the orientation angle of the hoop, which slowly varies from 0 to 2π. The particle motion in the plane with coordinates $\boldsymbol{q} = (x, y)$ can be described by the Hamiltonian[24]

$$H(\boldsymbol{q},\boldsymbol{p};X(t)) = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + V[x\cos(X(t)) + y\sin(X(t)), - x\sin(X(t)) + y\cos(X(t))], \quad (31)$$

where V is a confining potential which is zero in a narrow strip centred on the hoop and very large elsewhere. The rotator eigenstates for fixed X are[24]

$$\psi_n(\boldsymbol{q};X) = (a(\boldsymbol{q};X)/L^{\frac{1}{2}})\exp[2\pi i n s(\boldsymbol{q};X)/L], \quad (32)$$

where a is the amplitude which confines particle to the hoop. L is the length of the hoop and s is the arc length measured relative to an arbitrary material point. The adiabatic phase η_n, accumulated during one rotation of the hoop in commutative case is given by[24]

$$\eta_n = -8\pi^2 nA/L^2, \quad (33)$$

where A is the area of the hoop. Now we study the same problem in the noncommutative case. We first study the case of a circular hoop. In this case, the correction to Berry's phase due to the noncommutativity of space to the first order is given by the integral:

$$\frac{2\pi q}{L^2}\sum_{m\neq n} C_{mn}^1\int_0^{2\pi}\mathrm{d}X\int_0^L \mathrm{d}s \cdot\Big[n\mathrm{e}^{\frac{2\pi\mathrm{i}(m-n)s}{L}} + m\mathrm{e}^{\frac{2\pi\mathrm{i}(n-m)s}{L}}\Big] = 0. \quad (34)$$

This means that the correction for the circular hoop vanishes. For the case of a non-circular hoop on noncommutative space, the C_{mn}^1 coefficients are

$$\theta C_{mn}^1 = \frac{\langle\phi_m| - \frac{1}{2}\theta_{ij}\frac{\partial V}{\partial x_i}p_j|\phi_n\rangle}{E_n^0 - E_m^0}, \quad (35)$$

which depends on the form of the potential $V(x, y)$. The correction to Berry's phase is given by

$$\Delta\eta_n = \frac{4\pi^2}{L^2}\sum_{m\neq n} C_{mn}^{(1)}\int_0^L (m\mathrm{e}^{\frac{2\pi\mathrm{i}(n-m)s}{L}} + n\mathrm{e}^{\frac{-2\pi\mathrm{i}(n-m)s}{L}})q\sin(\alpha)\mathrm{d}s, \quad (36)$$

where α is the angle between the radius $\boldsymbol{q}$ and the tangent at each point. As mentioned above, this correction depends on the form of the potential function.

In conclusion, we have presented Berry's phase within the formulated noncommutative quantum mechanics. If there exists any noncommutativity of space–time in nature, as seems to emerge from different theories and arguments, its implications should appear in Berry's phase of physical systems such as the one treated here.

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