Green's Function Method for Perturbed Korteweg–de Vries Equation *

CAI Hao(蔡浩), HUANG Nian-Ning(黄念宁)
Department of Physics, Wuhan University, Wuhan 430072

(Received 2 December 2002)

The x-derivatives of squared Jost solution are the eigenfunctions with the zero eigenvalue of the linearized equation derived from the perturbed Korteweg–de Vries equation. A method similar to Green's function formalism is introduced to show the completeness of the squared Jost solutions in multi-soliton cases. It is not related to Lax equations directly, and thus it is beneficial to deal with the nonlinear equations with complicated Lax pair.

PACS: 05. 45. Yv, 02. 30. −f, 11. 10. Ef

A quasi-integrable nonlinear equation was treated either by the perturbation method based upon the inverse scattering transform[1,2] or by the direct perturbation method[3] (also see Ref. [4] and references therein). The latter is somewhat similar to that used in the perturbed linear equation. Its key point is to show the functions of expansion to form a complete set of basis.

The direct method for treating the perturbed Korteweg–de Vries (KdV) equation was developed in two stages. In the first stage, the unperturbed solution is restricted to a single soliton solution and the linearized equation is solved by separation of variables, for example, in Ref. [5] (also see Refs. [6,7] and references therein). Completeness of the set of eigenfunctions chosen is shown by using the Jordon theorem.[6] The procedure is obviously impossible to be extended to multi-soliton cases. In the second stage, by using the results of the inverse scattering transform, one can construct the so-called squared Jost solutions, and the $x-$derivatives of them are solutions of the linearized equation in multi-soliton cases. The completeness of them is shown recently. Using the Marchenko equation, the completeness of Jost solution is given in inverse scattering transform. Then a generalized Marchenko equation is derived and the completeness of the $x-$derivatives of the squared Jost solutions is shown[8] (also see Ref. [9] and references therein). This procedure is laborious, and is impossible to be extended to the perturbed nonlinear equation with a complicated Lax pair, such as the derivative nonlinear Schrödinger (DNLS) equation.

Recently, we paid attention to the papers given by Mann,[10,11] in which he also restricted to discuss a single soliton case. However, he has shown the completeness of solutions of the linearized equation by a method that is quite similar to that known as Green's function method used in the perturbed linear equation.[12] Since the completeness of solutions of the linearized equation is a linear problem, the procedure used in linear equation is also valid in the linearized equation. The method depends only on the linearized operator L and its adjoint operator L^A, but does not relate to Lax equations directly so that the complexity of Lax pair does not affect the procedure. It may be a prospective method. We now develop the procedure for proving completeness of x-derivatives of full squared Jost solution to the multi-soliton case of the perturbed KdV equation.

The perturbed KdV equation can be written in the form

$$v_t - 6vv_x + v_{xxx} = \epsilon r[v], \tag{1}$$

where $r[v]$ is a functional of v and ϵ is a small parameter. The initial condition is assumed to be $v(x,0) = u(x,0)$, where $u(x,0)$ is an exact soliton solution. In the first-order approximation, the solution of Eq. (1) under the initial condition can be expressed as

$$v(x,t) = u^a(x,t) + \epsilon q(x,t), \tag{2}$$

where $u^a(x,t)$ is the so-called adiabatic solution, which has the same functional form as that of the exact soliton solution but the parameters involved may depend on t of the order of ϵ and ϵq represents the remaining term which is also of the order of ϵ.

Substituting Eq. (2) into Eq. (1), a linearized equation is given by

$$Lq = R[u], \tag{3}$$

where the linearized operator L is introduced

$$L = \partial_t - 6\partial_x u^a + \partial_x^3, \tag{4}$$

and $R[u]$ is given by

$$R[u] = r[u] - s[u], \quad s[u] = \frac{1}{\epsilon}\{u_t - 6uu_x + u_{xxx}\}. \tag{5}$$

It should be noted that u in $r[u]$ is an exact soliton solution of the perturbed KdV equation, while u in $s[u]$ is the adiabatic solution (the superscript a has been dropped).

* Supported by the National Natural Science Foundation of China under Grant No 10071057.
©2003 Chinese Physical Society and IOP Publishing Ltd

The Lax equations of the unperturbed KdV equation are

$$\Big\{-\frac{\partial^2}{\partial x^2}+u\Big\}w(x,t;k)=k^2w(x,t;k), \tag{6}$$

$$\partial_t w(x,t;k)=\Big\{-4\frac{\partial^3}{\partial x^3}+6u\frac{\partial}{\partial x}+3u_x\Big\}w(x,t;k), \tag{7}$$

where k is the spectrum parameter, and $w(x,t;k)$ is the full Jost solution since it satisfies both Lax equations. The usual Jost solutions are defined as the solution of only the first Lax equation with suitable boundary behaviour, the full Jost solutions were obtained by including the time factor determined by the second Lax equation. It can be shown the x-derivatives of full squared Jost solution satisfy linearized equation

$$LW(x,t;k)_x=0, \tag{8}$$

where the full squared Jost solution

$$W(x,t;k)=w^2(x,t;k), \tag{9}$$

which means that the x-derivatives of full squared Jost solution are the eigenfunctions of the linearized equation with a zero eigenvalue.

Four full Jost solutions $\tilde{\psi}(x,t;k)$, $\psi(x,t;k)$, $\phi(x,t;k)$ and $\tilde{\phi}(x,t;k)$ are defined by their asymptotic behaviour

$$\begin{aligned}&\psi(x,t;k)\to \mathrm{e}^{\mathrm{i}4k^3t}\mathrm{e}^{\mathrm{i}kx}\\ &\tilde{\psi}(x,t;k)\to \mathrm{e}^{-\mathrm{i}4k^3t}\mathrm{e}^{-\mathrm{i}kx}\quad \text{as}\quad x\to\infty,\end{aligned} \tag{10}$$

$$\begin{aligned}&\phi(x,t;k)\to \mathrm{e}^{-\mathrm{i}4k^3t}\mathrm{e}^{-\mathrm{i}kx}\\ &\tilde{\phi}(x,t;k)\to \mathrm{e}^{\mathrm{i}4k^3t}\mathrm{e}^{\mathrm{i}kx}\quad \text{as } x\to-\infty.\end{aligned} \tag{11}$$

We thus have four x-derivatives of full squared Jost solution, for example,

$$\Psi(x,t;k)=\psi^2(x,t;k),\ldots. \tag{12}$$

An operator L^A, the adjoint of L, is defined by

$$gLf-fL^Ag=\text{divergence}, \tag{13}$$

where f and g are any two functions, the adjoint linearized operator L^A is obtained simply by partial integration

$$L^A=-\partial_t+6u^a\partial_x-\partial_x^3. \tag{14}$$

It can be shown

$$L^AW(x,t;k)=0 \tag{15}$$

which means adjoint linearized function is full squared solutions, $\tilde{\Psi}(x,t;k)$, $\Psi(x,t;k)$, $\Phi(x,t;k)$ and $\tilde{\Phi}(x,t;k)$.

The inhomogeneous Eq. (3) is formally solved by Green's function $G(x,t;x',t')$ defined by

$$LG=\delta(x-x')\delta(t-t'), \tag{16}$$

and its solution may be expressed as

$$q=\int_0^\infty \mathrm{d}t'\int_{-\infty}^{\infty}\mathrm{d}x'G(x,t;x',t')R(x,t'), \tag{17}$$

which is easily verified by forming Lq. Because of the causality condition, G should be zero for $t'>t$. This is achieved

$$G=G^0(x,t;x',t')\theta(t-t'), \tag{18}$$

where $\theta(\tau)$ designates the step function, $\theta(\tau)=1$ for $\tau>0$, $\theta(\tau)=0$ for $\tau<0$. Then the solution Eq. (17) becomes

$$q=\int_0^t \mathrm{d}t'\int_{-\infty}^{\infty}\mathrm{d}x'G^0(x,t;x',t')R(x,t'). \tag{19}$$

This solution fulfils the initial condition $q=0$ for $t=0$. Inserting Eq. (19) into Eq. (3) gives

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x'G^0(x,t;x',t)R(x',t)\\ &\qquad+\int_0^t\mathrm{d}t'\int_{-\infty}^{\infty}\mathrm{d}x'LG^0(x,t;x',t')R(x',t')\\ &\qquad=R(x,t).\end{aligned} \tag{20}$$

This equation is identically satisfied by assuming that G^0 is a solution of the homogeneous equation

$$LG^0(x,t;x',t')=0 \tag{21}$$

and obeys the final condition

$$G^0(x,t;x',t')|_{t=t'}=\delta(x-x'). \tag{22}$$

These two equations define a final-value problem for Green's function G^0.

$\Psi(x,t;k)_x$ is a solution of the linearized Eq. (8) where continuous spectrum is real k. It is analytical in the upper half of k-plane, so that $\Psi(x,t;k_n)_x$ is a solution for discrete spectrum k_n in the upper imaginary axis. Moreover, the fact that there is no spectrum parameter in the linearized operator L gives that $\dot{\Psi}(x,t;k_n)_x$ is also a solution, where $\dot{\Psi}(x,t;k_n)_x=\dfrac{\mathrm{d}}{\mathrm{d}k}\Psi(x,t;k)_x\Big|_{k=k_n}$. Thus we can expand Green's function $G^0(x,t;x',t')$ as a linear combination of discrete and continuous basis functions

$$\begin{aligned}G^0(x,t;x',t')=&\sum_n A(x',t';k_n)\,\Psi(x,t;k_n)_x\\ &+\sum_n B(x',t';k_n)\,\dot{\Psi}(x,t;k_n)_x\\ &+\int_\Gamma \mathrm{d}k A(x',t';k)\,\Psi(x,t;k)_x,\end{aligned} \tag{23}$$

and Eq. (22) becomes

$$\begin{aligned}\delta(x-x') =& \sum_n A(x',t;k_n)\,\Psi(x,t';k_n)_x \\ &+ \sum_n B(x',t;k_n)\,\dot{\Psi}(x,t;k_n)_x \\ &+ \int_\Gamma \mathrm{d}k A(x',t;k)\,\Psi(x,t;k)_x,\end{aligned} \tag{24}$$

where the integral path Γ is along the real axis from $-\infty$ to $+\infty$, but goes over the origin in the upper k-plane.

Multiplying Eq. (24) by $\Phi(x,t;k')$ and integrating over x yields

$$\begin{aligned}\Phi(x',t;k') &= \int_\Gamma \mathrm{d}k A(x',t;k) M(t;k',k), \\ M(t;k',k) &= \int_{-\infty}^{\infty} \mathrm{d}x\,\Phi(x,t;k')\,\Psi(x,t;k)_x.\end{aligned} \tag{25}$$

Since $L\,\Psi(x,t;k)_x = 0$ and $L^A\,\Phi(x,t;k') = 0$, replacing f and g in Eq. (13) by $\Psi(x,t;k)_x$ and $\Phi(x,t;k)$ respectively, we have

$$\begin{aligned}0 =& \frac{\partial}{\partial t}[\Phi(x,t;k')\,\Psi(x,t;k)_x] \\ &+ \frac{\partial}{\partial x}[6u_0\,\Psi(x,t;k')_x\,\Phi(x,t;k) \\ &+ \Psi(x,t;k')_{xxx}\,\Phi(x,t;k) + \Psi(x,t;k')_x\,\Phi(x,t;k)_{xx} \\ &- \Psi(x,t;k')_{xx}\,\Phi(x,t;k)_x].\end{aligned} \tag{26}$$

Integration over x provides a relationship for the elements $M(t;k',k)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t;k',k) =& -[\,\Psi(x,t;k')_{xxx}\,\Phi(x,t;k) \\ &+ \Psi(x,t;k')_x\,\Phi(x,t;k)_{xx} \\ &- \Psi(x,t;k')_{xx}\,\Phi(x,t;k)_x]\big|_{x=-\infty}^{x=\infty}.\end{aligned} \tag{27}$$

From the asymptotic behaviours Eq. (10) and Eq. (11), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t;k',k) =& \lim_{L\to\infty} \mathrm{i}8k'(k^2+kk'+k'^2) \\ &\cdot \mathrm{e}^{\mathrm{i}8(k'^3-k^3)t}[a(k)^2\mathrm{e}^{\mathrm{i}2(k'-k)L} - a(k')^2 \\ &\cdot \mathrm{e}^{\mathrm{i}2(k'-k)L}],\end{aligned} \tag{28}$$

and then

$$\begin{aligned}(k'-k)M(t;k',k) =& \lim_{L\to\infty} k'\mathrm{e}^{\mathrm{i}8(k'^3-k^3)t}[a(k)^2 \\ &\cdot \mathrm{e}^{\mathrm{i}2(k'-k)L} - a(k')^2\mathrm{e}^{\mathrm{i}2(k'-k)L}],\end{aligned} \tag{29}$$

where $a(k)$ is the element of the monodramy matrix for the KdV equation. Since $\Psi(x,t;k)$ and $\Phi(x,t;k)$ are analytical in the upper half k-plane, we can deal with the limitation as the principal value. Noticing the formula

$$\lim_{L\to\infty} P\frac{1}{k'-k}\mathrm{e}^{\mathrm{i}(k'-k)L} = \mathrm{i}\pi\delta(k'-k), \tag{30}$$

where P denotes the principal value, we obtain

$$M(t;k',k) = \mathrm{i}2\pi a(k)^2 k\delta(k'-k). \tag{31}$$

Introducing this result into Eq. (25), the coefficient $A(x',t;k)$ is obtained to be

$$A(x',t;k) = \frac{1}{\mathrm{i}2\pi a(k)^2}\,\Phi(x',t;k). \tag{32}$$

Then multiplying Eq. (24) by $\Phi(x,t;k_m)$, we obtain

$$\begin{aligned}\Phi(x',t;k_m) =& \sum_n A(x',t;k_n)M^{AA}(t;k_m,k_n) \\ &+ \sum_n B(x',t;k_n)M^{AB}(t;k_m,k_n),\end{aligned} \tag{33}$$

where

$$M^{AA}(t;k_m,k_n) = \int_{-\infty}^{\infty} \mathrm{d}x\,\Phi(x,t;k_m)\,\Psi(x,t;k_n)_x, \tag{34}$$

$$M^{AB}(t;k_m,k_n) = \int_{-\infty}^{\infty} \mathrm{d}x\,\Phi(x,t;k_m)\,\dot{\Psi}(x,t;k_n)_x. \tag{35}$$

Since $a(k_m) = a(k_n) = 0$, it is easy to know

$$M^{AA}(t;k_m,k_n) = 0. \tag{36}$$

From Eq. (29), we have

$$\begin{aligned}M^{AB}(t;k_m,k_n) =& -\frac{\mathrm{d}^2}{\mathrm{d}k^2}(k'-k)M(t;k',k)\Big|_{k'=k_n,k=k_m} \\ =& k_n\dot{a}(k_n)^2\delta_{mn},\end{aligned} \tag{37}$$

and

$$B(x',t;k_n) = \frac{1}{k_n\dot{a}(k_n)^2}\,\Phi(x',t;k_n). \tag{38}$$

Finally, multiplying Eq. (24) by $\dot{\Phi}(x,t;k_m)$, we obtain

$$\begin{aligned}\dot{\Phi}(x',t;k_m) =& \sum_n A(x',t;k_n)M^{BA}(x';k_m,k_n) \\ &+ \sum_n B(x',t;k_n)M^{BB}(x';k_m,k_n),\end{aligned} \tag{39}$$

where

$$M^{BA}(x';k_m,k_n) = \int_{-\infty}^{\infty} \mathrm{d}x\,\dot{\Phi}(x,t;k_m)\,\Psi(x,t';k_n)_x, \tag{40}$$

$$M^{BB}(x';k_m,k_n) = \int_{-\infty}^{\infty} \mathrm{d}x\,\dot{\Phi}(x,t;k_m)\,\dot{\Psi}(x,t;k_n)_x. \tag{41}$$

With the same method used in Eq. (37), we have

$$M^{BA}(x';k_m,k_n)=M^{AB}(x';k_m,k_n)=k_n\dot{a}(k_n)^2\delta_{mn}. \tag{42}$$

Then applying the operator

$$\frac{\mathrm{d}^3}{\mathrm{d}k^3}+3\frac{\mathrm{d}}{\mathrm{d}k'}\frac{\mathrm{d}^2}{\mathrm{d}k^2}$$

to Eq. (29) and then taking $k'=k_m$ and $k=k_n$, we obtain

$$M^{BB}(x';k_m,k_n)=\dot{a}(k_n)\ddot{a}(k_n)k_n\delta_{mn}+\dot{a}(k_n)^2\delta_{mn}. \tag{43}$$

As a result, we obtain the equation

$$\begin{aligned}\dot{\Phi}(x',t;k_m)=&A(x',t;k_n)k_n\dot{a}(k_n)^2\\&+\left\{\frac{\ddot{a}(k_n)}{\dot{a}(k_n)}+\frac{1}{k_n}\right\}\Phi(x',t;k_n),\end{aligned} \tag{44}$$

that is

$$\begin{aligned}A(x',t;k_n)=&\frac{1}{k_n\dot{a}(k_n)^2}\dot{\Phi}(x',t;k_m)\\&-\frac{1}{k_n\dot{a}(k_n)^2}\left\{\frac{\ddot{a}(k_n)}{\dot{a}(k_n)}+\frac{1}{k_n}\right\}\Phi(x',t;k_n).\end{aligned} \tag{45}$$

Hence we prove the completeness of x-derivatives of full squared Jost solutions to the multi-soliton case of the perturbed KdV equation

$$\begin{aligned}\delta(x-x')=&-\sum_n\frac{1}{k_n\dot{a}(k_n)^2}\\&\cdot\left\{\frac{\ddot{a}(k_n)}{\dot{a}(k_n)}+\frac{1}{k_n}\right\}\Psi(x,t';k_n)_x\Phi(x',t;k_n)\\&+\sum_n\frac{1}{k_n\dot{a}(k_n)^2}\Big\{\Psi(x,t';k_n)_x\dot{\Phi}(x',t;k_m)\\&+\dot{\Psi}(x,t;k_n)_x\Phi(x',t;k_n)\Big\}\\&+\int_\Gamma\mathrm{d}k\frac{1}{i2\pi ka(k)^2}\Psi(x,t;k)_x\Phi(x',t;k).\end{aligned} \tag{46}$$

In conclusion, the x-derivatives of squared Jost solution are the eigenfunctions with zero eigenvalue of the linearized equation with operator L in multi-soliton case, while the squared Jost solutions are the eigenfunctions of the adjoint linearized equation with operator L^A. In the first order of ϵ, the problem is essentially a linear one, which means it is reasonable to apply usual Green's function formulation[12] to show the completeness of the x-derivative squared Jost solutions. Since the two linearized operators L and L^A do not relate to Lax equations directly, the complexity of Lax pair is impossible to affect the procedure of the proof so that this method can be used to deal with other perturbed nonlinear equations with complicated Lax pairs.

References

[1] Kaup D J 1976 *SIAM J. Appl. Math.* **31** 121
[2] Karpman V I 1977 *JETP Lett.* **25** 271
[3] Kaup D J 1983 *Phys. Rev.* B **27** 6787
[4] Herman R L 1990 *J. Phys. A: Math. Gen.* **23** 2371
[5] Otto E and Sudan R N 1970 *Phys. Fluids* **13** 1432
[6] Yan J R and Tang Y 1996 *Phys. Rev.* E **54** 6816
[7] Cai H, Shi J, Tian D C and Huang N N 2002 *Chin. Phys. Lett.* **19** 908
[8] Moses H E 1977 *J. Math. Phys.* **18** 2243
[9] Chen Z D, Chen S R, Huang N N 1998 *Acta Phys. Sin.* **48** 887 (in Chinese)
[10] Mann E 1997 *J. Math. Phys.* **38** 3772
[11] Mann E 1997 *J. Phys. A: Math. Gen.* **30** 1227
[12] Morse P M, Feshbach H 1953 *Methods of Theoretical Physics* (New York: Mc Graw-Hill)