Walks on Weighted Networks *

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We investigate the dynamics of random walks on weighted networks. Assuming that the edge weight and the node strength are used as local information by a random walker. Two kinds of walks, weight-dependent walk and strength-dependent walk, are studied. Exact expressions for stationary distribution and average return time are derived and confirmed by computer simulations. The distribution of average return time and the mean-square displacement are calculated for two walks on the Barrat–Barthélemy–Vespignani (BBV) networks. It is found that a weight-dependent walker can arrive at a new territory more easily than a strength-dependent one.

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There has been a long history of studying random walks to model various dynamics in physical, biological, social, and economic systems.[1] A large body of theoretical results are available for random walks performed on regular lattices and on the Cayley (or regular) trees,[2,3] which are defined to be trees with homogeneous vertex degree. However, it has been suggested recently that more complex networks as opposed to regular graphs and conventional random graphs[4] are concerned to real worlds. Particularly, important classes of random graphs such as small-world networks[5] and scale-free networks[6] were proposed and have been examined in the last several years. These networks share some important properties with real networks, such as the clustering property, short average path length, and the power-law of the vertex degree distributions. They have been applied to the analysis of various social, engineering, and biological networks including epidemic spreading, percolation, and synchronization, and so on.[7−10]

Recently, there have been several studies of random walks on small-world networks[11−14] and on scale-free networks.[15−18] Most of the studies of random walks focus on unweighted networks, however the study of the dynamics of random walks on weighted networks is missing while most of real networks are weighted characterized by capacities or strengths instead of a binary state (present or absent).[19−22] In weighted networks, a weight w_{ij} is assigned to the edge connecting the vertices i and j, such as in the airport network, each given weight w_{ij} is the number of available seats on direct flight connections between the airports i and j. Along with the degree of a node that is the number of connected neighbours, the strength of the vertex i in weighted networks can be defined as[19]

$$s_i = \sum_{j\in\nu(i)} w_{ij}, \tag{1}$$

where the sum runs over the set $\nu(i)$ of neighbours of i. The strength of a node integrates the information about its connectivity and the weights of its links.

In this Letter, we study the dynamics of random walk processes on weighted networks by means of the BBV model which is one of the most well-known models introduced by Barrat, Barthélemy, and Vespignani.[19] The model starts from an initial number of completely connected vertices m_0 with a same assigned weight w_0 to each link. At each subsequent time step, addition of a new vertex n with m_0 edges and the corresponding modification in weights are implemented by the following two rules: (i) The new vertex n is attached at random to a previously existing vertex i with the probability that is proportional to the strength of node i, $s_i/\sum_j s_j$, implying that new vertices connect more likely to vertices handling larger weights. (ii) The additional induced increase δ in strength s_i of the ith vertex is distributed among its nearest neighbours $j \in \mathcal{V}(i)$ according to the rule

$$w_{ij} \to w_{ij} + \delta\frac{w_{ij}}{s_i}, \tag{2}$$

which considers that the establishment of a new edge of weight w_0 with the vertex i induces a total increase of traffic δ that is proportionally distributed among the edges departing from the vertex according to their weights. The BBV model suggests two ingredients of self-organization of weighted networks, strength preferential attachment and weight evolving dynamics.[19] Consider an arbitrary finite weighted network which consists of nodes $i = 1, \cdots, N$ and links connecting them. The connectivity is represented by the adjacency matrix $\boldsymbol{A}$ whose element $a_{ij} = 1$ if there is a link from i to j, and $a_{ij} = 0$ otherwise. The information of edges weight is represented by matrix $\boldsymbol{W}$ whose element w_{ij} is the weight of the edge between i and j. We restrict ourselves to an undirected network

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$a_{ij} = a_{ji}$ and symmetrical edge's weight $w_{ij} = w_{ji}$.

Assuming that edge's weight and node's strength are used as local information by a random walker, we define two kinds of walks, weight-dependent walk and strength-dependent walk. For weight-dependent walk, a walker chooses one of its nearest neighbours with the probability that is proportional to the weight of edge linked them. The transition probability from node i to its neighbour j is

$$p^w_{i\to j} = \frac{w_{ij}}{s_i}. \tag{3}$$

When time becomes infinite, one can find the walker staying at node i with the probability P^w_i, which is defined as the stationary distribution. Noh and Rieger[16] have given the result:

$$P^w_i = \frac{s_i}{\sum_j s_j}, \tag{4}$$

namely, the larger strength a node has, the more frequently it will be visited by a random walker.

For strength-dependent walk, a walker at node i at time t selects one of its neighbours with the probability which is proportional to the selected node's strength to which it hops at time $t+1$. The transition probability from node i to its neighbour j is

$$P^s_{i\to j} = \frac{s_j a_{ij}}{s'_i}, \tag{5}$$

where $s'_i = \sum_{l\in\nu(i)} s_l$ and $\nu(i)$ denotes the set of all neighbouring vertices of node i. Supposing that a walker starts at node i, and the probability that the walker at node k after t time steps denoted by $P^s_{ik}(t)$, then the master equation for the probability $P^s_{i\to j}$ to find the walker at node j at time $t+1$ is

$$P^s_{i\to j}(t+1) = \sum_k \frac{s_j a_{kj}}{s'_k} P^s_{ik}(t). \tag{6}$$

An explicit expression for the transition probability $P^s_{i\to j}(t)$ to go from node i to node j in t steps follows by iterating Eq. (6)

$$P^s_{i\to j}(t) = \sum_{j_1,\cdots,j_{t-1}} \frac{s_{j_1}a_{ij_1}}{s'_i}\cdot\frac{s_{j_2}a_{j_1j_2}}{s'_{j_1}}\cdots\frac{s_j a_{j_{t-1}j}}{s'_{j_{t-1}}}. \tag{7}$$

Comparing the expressions for $P^s_{i\to j}(t)$ and $P^s_{j\to i}(t)$, we obtain

$$s_i s'_i P^s_{i\to j}(t) = s_j s'_j P^s_{j\to i}(t). \tag{8}$$

This is a direct consequence of the undirectedness of the network. We can also define the probability P^s_i as the stationary distribution when the evolving time becomes infinite. Equation (8) implies that $s_i s'_i P^s_j = s_j s'_j P^s_i$, and therefore one can obtain

$$P^s_i = \frac{s_i s'_i}{\sum_l s_l s'_l}. \tag{9}$$

Now we discuss the stationary distribution for edges, i.e. the probability that an edge is chosen by the walker to follow as the evolving time becomes infinite. In unweighted networks, all edges are equal and a random walker will choose one of its neighbouring edges at the same probability. Thus each edge in the network has the same probability to be chosen by the walker when $t\to\infty$ for weight-free walk (standard random walk). In weighted networks, however, the walker will choose an edge according to its weight or the strength of the node connected by it. Then the relation between the stationary distribution for edges $P_{e_{ij}}$ and the stationary distribution for nodes P_i can be written as

$$P_{e_{ij}} = P_i P_{i\to j} + P_j P_{j\to i}, \tag{10}$$

where $P_{i\to j}$ is the transition probability from node i to node j. The term $P_i P_{i\to j}$ presents node ith contribution to the stationary distribution for edges $P_{e_{ij}}$, in other words, it is the stationary distribution for the directed edge e_{ij} in directed networks.[23] For weight-dependent walk, substituting Eqs. (3) and (4) into Eq. (10), we obtain the stationary distribution for edges,

$$P^w_{e_{ij}} = \frac{2w_{ij}}{\sum_{k,l} w_{kl}}. \tag{11}$$

For strength-dependent walk, substituting Eqs. (5) and (9) into Eq. (10), we obtain the stationary distribution for edges,

$$P^s_{e_{ij}} = \frac{2s_i s_j}{\sum_{k,l} s_k s_l a_{kl}}. \tag{12}$$

Figures 1(a) and 1(b) show P^w_i vs s_i and $P^w_{e_{ij}}$ vs w_{ij}, respectively, in the log–log scale in the BBV network. The power-law property of Eqs. (4) and (11) is presented in excellent agreement with the numerical results. Figures 2(a) and 2(b) show the log–log plots of P^s_i vs $s_i s'_i$ and $P^s_{e_{ij}}$ vs $s_i s_j$ in the BBV model. The numerical results are also in good agreement with Eqs. (9) and (12).

Next we study average return time which is the average time spent by a walker to return to its origin. From its definition, we can easily obtain that the average return time is equal to the reciprocal of the stationary distribution. The average return time for node i is

$$\langle T^w_{ii}\rangle = \frac{1}{P^w_i} = \frac{\sum_j s_j}{s_i}, \tag{13}$$

for weight-dependent walk and

$$\langle T^s_{ii}\rangle = \frac{1}{P^s_i} = \frac{\sum_j s_j s'_j}{s_i s'_i}. \tag{14}$$

for strength-dependent walk, respectively.

Using the same methods, the average return time for edge e_{ij} can also be obtained to be

$$\langle T^w_{e_{ij}}\rangle = \frac{\sum_{k,l} w_{kl}}{2w_{ij}}, \tag{15}$$

for weight-dependent walk and

$$\langle T^s_{e_{ij}} \rangle = \frac{\sum_{k,l} s_k s_l a_{kl}}{2 s_i s_j}. \qquad (16)$$

for strength-dependent walk, respectively.

Figure 3 shows the log–log plots of $\langle T^w_{ii} \rangle$ vs $s(i)$ and $\langle T^s_{ii} \rangle$ vs $s(i)$ in the BBV networks. The slope of the curve in Fig. 3(a) is -0.9903 ± 0.0007, consistent with Eq. (13). The BBV networks have two properties described in Ref. [19]: (i) node strength is proportional to node degree, $s_i \sim k_i$; (ii) the average nearest neighbour degree is $k_{nn}(k) \sim k^{-2+1/\beta}$ with $\beta = (2\delta + 1)/(2\delta + 2)$. Considering these two ingredients, one can observe that $s_i s'_i \sim s_i^{1/\beta}$, and obtain $\langle T^s_{ii} \rangle \sim s_i^{-1/\beta}$. Figure 3(b) gives that $-1/\beta = -1.355 \pm 0.006$, which agrees with the theoretical value $-1/\beta = -(2\delta + 2)/(2\delta + 1) = -4/3$. In Fig. 3, the slope of the curve for strength-dependent walk is steeper than that for weight-dependent walk, giving rise to a broader distribution of average return time for strength-dependent walk. In order to confirm this point, we derive the expression for the distribution of average return time. In BBV networks, we know that the strength distribution behaves as $P(S) \sim S^{-\gamma}$, where $\gamma = (3+4\delta)/(1+2\delta)$.[19] According to the above analysis, we can obtain that the distribution of average return time is $P(T^w) \sim (T^w)^{-1/(1+\delta)}$ for the weight-dependent walk, and $P(T^s) \sim 1.0$ for the strength-dependent case. Thus, we can see that for those nodes with large strength, the strength-dependent walker spends more time in visiting them than that spent by the weight-dependent walker. This point can be reflected by the difference of the mean-square displacement for two walks which is shown in the following.

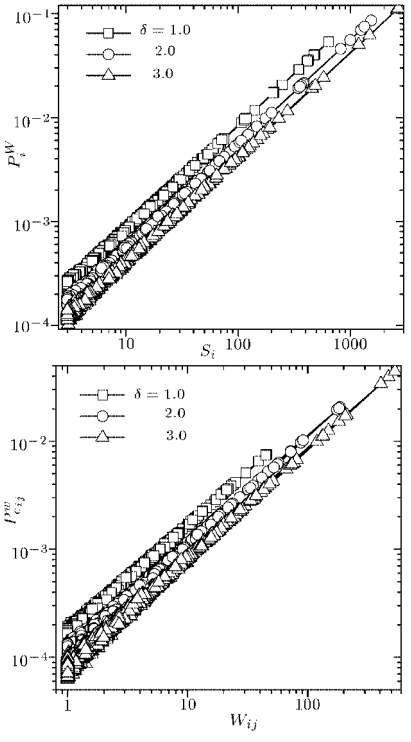


Fig. 1. Log–log plots of P_i^w vs s_i (a) and $P^w_{e_{ij}}$ vs w_{ij} (b) in the BBV network with $N = 1000$, $m_0 = 3$ and different values of δ. Slopes of all the curves are equal to 0.9999 ± 0.0006. The data were obtained after walking 10^8 steps on the network.

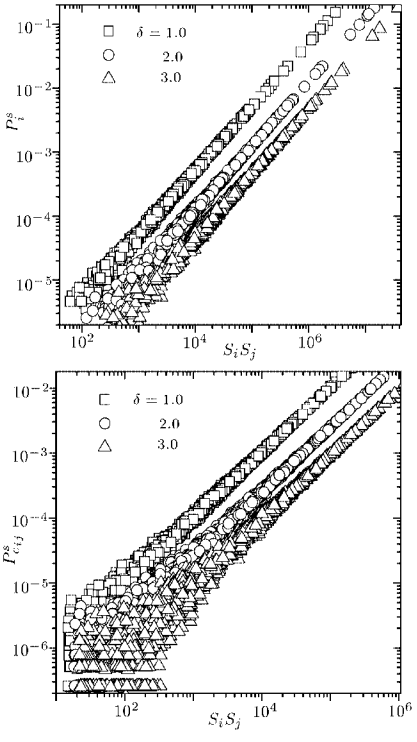


Fig. 2. Log–log plots of P_i^s vs $s_i s'_i$ (a) and $P^s_{e_{ij}}$ vs $s_i s_j$ (b) in the BBV network with $N = 1000$, $m_0 = 3$ and different values of δ. Slopes of all the curves are equal to 1.018 ± 0.003. The data were obtained after walking 10^8 steps on the network.

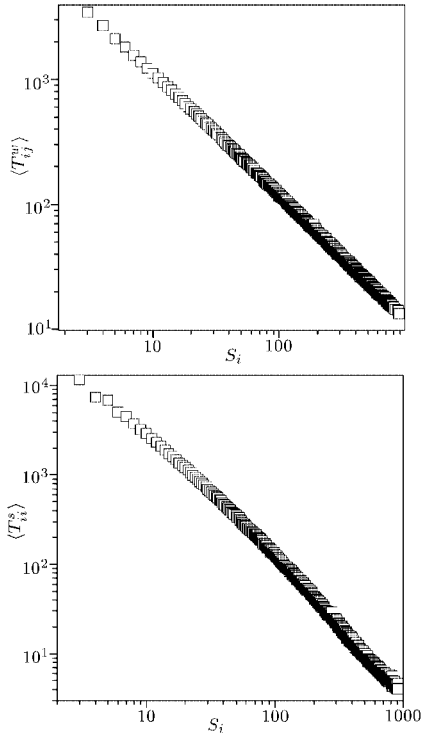


Fig. 3. Numerical results for the average return time of node for the two different walks, (a) weight-dependent walk and (b) strength-dependent walk, on the BBV network with $N = 1000$, $m_0 = 3$ and $\delta = 1.0$. The data are averaged on 1000 networks and obtained after walking 10^6 steps in each network.

The mean-square displacement $\langle R^2 \rangle$ is a measure of the distance R covered by a typical random walker after performing t steps.[13,17] To calculate this quantity we first, at each time step, find the minimal distance from the current position of the walker to the origin (i.e. the smallest number of steps needed for the walker to reach the origin) using a breadth-first search method. Then we allow the walker to move through the network until $\langle R^2 \rangle$ has saturated. Finally, we average over different initial positions of the walker and realizations of the network. Figure 4 shows the results of $\langle R^2 \rangle$ as a function of time t. We note that $\langle R^2 \rangle$ equilibrates after a few steps to a constant displacement value. This is a simple manifestation of the very small

diameter of the BBV network. It is also noted that the value of plateau is higher for weight-dependent walk than that for strength-dependent walk. That is, a strength-dependent walker spends more time in visiting these nodes with large strength or large degree, i.e. a weight-dependent walker can arrive at a new territory more easily than a strength-dependent walker. It is notable that for weight-free walk, the stationary distribution is proportional to the node degree,[16] and in the BBV network the node degree is proportional to the node strength, weight-free walker and weight-dependent walker have the same probability for visiting a given node, and the values of the mean-square displacement are equal.

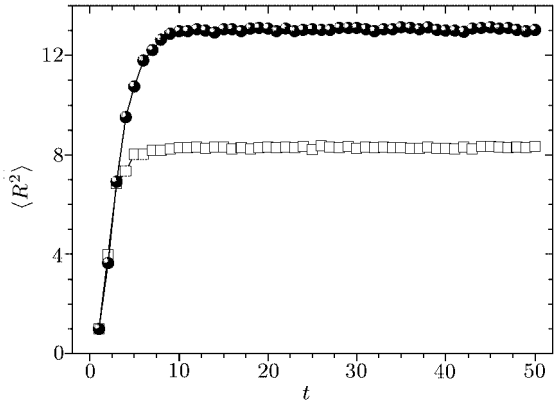


Fig. 4. Mean-square displacement $\langle R^2 \rangle$ as a function of time t for two kinds of walks, weight-dependent walk (circles) and strength-dependent walk (squares), on the BBV network with $N = 10^5$, $m_0 = 3$, and $\delta = 1.0$. All the plots are averaged over 1000 different initial positions of the walker and 20 realizations of the network.

In summary, we have studied two different types of walks on weighted networks, i.e. weight-dependent walk and strength-dependent walk. We have derived exact expressions for the stationary distribution and the average return time for the two walk processes, and confirmed them by simulations on BBV networks. Then we analysed the distribution of average time for two walks and found that it is broader for dependent-strength walk than that for dependent-weight walk on the BBV network. Finally the mean-square displacement $\langle R^2 \rangle$ is computed. For both the walks, $\langle R^2 \rangle$ is found to reach the saturation after a few time steps, which is a result of the very small diameter of the underlying graph. Furthermore, the difference of average-square displacement for two walks implies that a weight-dependent walker can arrive at a new territory more easily than a strength-dependent walker on the BBV network.

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