Synchronization of Complex Network with Drivingly Coupled Scheme *

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We present a network model with a new coupled scheme which is the generalization of drive-response systems called a drivingly coupled network. The synchronization of the network is investigated by numerical simulations based on Lorenz systems. By calculating the largest transversal Lyapunov exponents of such network, the stable and unstable regions of synchronous state for eigenvalues in such network can be obtained and many kinds of drivingly coupled arrays based on Lorenz systems such as all-to-all, star-shape, ring-shape and chain-shape networks are considered.

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Synchronization is a very common phenomenon in natural systems. Since the discovery of the small-world network[1] and scale-free network,[2] the synchronous problem of networks has been an active research topic in many fields ranging from physics and neurobiology to engineering and social sciences.[3−10] Wang and Chen[3,4] studied the synchronization of small-world network and showed the robustness and fragility of synchronization in scale-free network. Lü and Chen[5] studied the synchronization of a time-varying complex dynamical network models and obtained its controlled synchronization criteria. Zhou *et al.*[6] further investigated the locally and globally adaptive synchronization of an uncertain complex dynamical network.

Nowadays, most ways of coupling in general networks are on the basis of feedback coupling. However, the manners of exchanging signal in nature are various. Motivated by the this fact, we contribute to propose a model of network with a new coupled scheme and investigate its synchronization properties. The network with this coupling called drivingly coupled network is the extension of drive-response systems used to study the synchronization of two chaotic systems.

In 1990, Pecora and Carroll[7] presented the P-C method to successfully keep two chaotic systems synchronous. The basic idea of the P-C method is to divide a chaotic system into two subsystems, take one subsystem as a drive system and duplicate the other as a response system, then let the output signal of the drive system be the input signal of the response system to obtain the synchronization of the two systems. The stability of synchronization is determined by computing the transverse Lyapunov exponents of drive system and response system.[8−10] If all the transverse Lyapunov exponents are negative, the synchronized manifold is stable, otherwise it is unstable.

In this Letter, by the extending and using the P-C method, we present a network model with a driving coupling and investigate its stability of synchronous state. For describing the driving coupling, we consider the simple case of two systems,

$$\begin{cases} \dot{X}_1 = f_1(X_1, X_2), \\ \dot{X}_2 = f_{21}(X_1, Y_2) + f_{22}(X_1, X_2), \end{cases} \tag{1}$$

$$\begin{cases} \dot{Y}_1 = f_1(Y_1, Y_2), \\ \dot{Y}_2 = f_{21}(Y_1, X_2) + f_{22}(Y_1, Y_2), \end{cases} \tag{2}$$

where $X_1, Y_1 \in \mathbb{R}^p, X_2,\ Y_2 \in \mathbb{R}^q,\ p+q=d,\ f_{21}(\cdot)$ is the part of the subsystem involving the exchange of signal with other systems and it can be linear or nonlinear, $f_{22}(\cdot)$ is the part of subsystems and does not take part in the exchange of signal. Systems (1) and (2) are symmetric, namely, the quantity of exchange is equal. On the one hand, Y_2 is the input signal of system (1) and equally, on the other hand X_2 is the input signal of system (2), which is different from the drive-response systems by the P-C method.

In view that the dynamical behaviours of an individual system is always affected by all kinds of factors from its neighbours, it is more general for us to consider the drivingly coupled network,

$$\begin{aligned} \dot{X}_1^i &= f_1(X_1^i, X_2^i), \\ \dot{X}_2^i &= f_{21}\Big(X_1^i, \frac{1}{k_i}\sum_{j\neq i, j=1}^{n} g_{ij} X_2^j\Big) + f_{22}(X_1^i, X_2^i), \end{aligned} \tag{3}$$

where $X^i = (X_1^i, X_2^i) \in \mathbb{R}^d,\ i = 1, \ldots, n$ means the state variable of the ith system, k_i is the connectivity degree of node i, $k_i = \sum_{j\neq i, j=1}^{n} g_{ij}$ and g_{ij} is defined by $g_{ij} = 1$ if nodes i and j are connected, and $g_{ij} = 0$ otherwise.

In Eq. (3), the input signal $\frac{1}{k_i}\sum_{j\neq i, j=1}^{n} g_{ij} X_2^j$ of the ith system is just the average output signal from its connective systems. It is observed that the scheme

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of mutual drive can be taken as a mode of coupling. This coupling is called the driving coupling since the systems with this coupling exchange the signal with their neighbours. It looks like that one part of a system can be driven by the same part of other systems in this coupled network. Specifically, we consider the drivingly coupled network based on Lorenz systems in the following.

We employ n Lorenz systems to connect in a network with driving coupling and the equations of such network are described as

$$\begin{aligned}
\dot{x}_i &= \sigma(y_i - x_i), \\
\dot{y}_i &= \rho\Big(\frac{1}{k_i}\sum_{j\neq i, j=1}^{n} g_{ij}x_j\Big) - y_i - x_i z_i \\
\dot{z}_i &= x_i y_i - b z_i, \quad i = 1, 2, \cdots, n,
\end{aligned} \tag{4}$$

where σ, ρ and b are the parameters of Lorenz systems. Clearly, the coupled systems can be seen as drive-response systems and the drive term of the ith system is $\frac{1}{k_i}\sum_{j\neq i, j=1}^{n} g_{ij}x_j$. The coupling of networks can also be nonlinear, for example,

$$\begin{aligned}
\dot{x}_i &= \sigma(y_i - x_i), \\
\dot{y}_i &= \rho\Big(\frac{1}{k_i}\sum_{j\neq i, j=1}^{n} g_{ij}x_j\Big) - y_i - \Big(\frac{1}{k_i}\sum_{j\neq i, j=1}^{n} g_{ij}x_j\Big) z_i, \\
\dot{z}_i &= \Big(\frac{1}{k_i}\sum_{j\neq i, j=1}^{n} g_{ij}x_j\Big) y_i - b z_i, \quad i = 1, 2, \cdots, n.
\end{aligned} \tag{5}$$

In the following, we aim at investigating the synchronization of systems (4) by numerical simulation. Firstly, we generate a network-like B-A model (Barabási–Albert model).[2] Start from $t = 0$ with 2 nodes connected with an edge. At $t = 1$, add a new node and 2 edges linking that node to those existent nodes. Then at all times $t > 1$, add a new node and 2 edges connecting this node to those existing nodes by taking into account preferential attachment of a new node to an existing node. Considering the workload of simulating the synchronization of network, we end the evolution at time $t = 18$ and obtain a network of 20 nodes as shown in Fig. 1.

When the structure of network is established, we can clearly write the concrete form of systems (4) with the driving coupling. For node 1, which connects with nodes 2, 3, 4, 5, 6, 8, 17, 18, 19 and 20, the dynamical equations of node 1 can be written as

$$\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1), \\
\dot{y}_1 &= \frac{\rho}{10}(x_2 + x_3 + x_4 + x_5 + x_6 + x_8 \\
&\quad + x_{17} + x_{18} + x_{19} + x_{20}) - y_1 - x_1 z_1, \\
\dot{z}_1 &= x_1 y_1 - b z_1.
\end{aligned} \tag{6}$$

The other systems of the network also have the similar form as Eqs. (6).

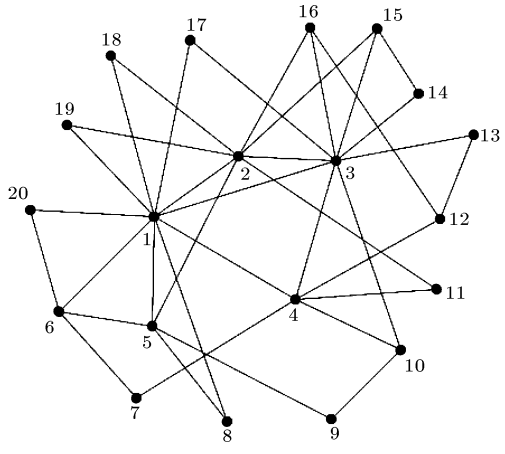


Fig. 1. B–A network.

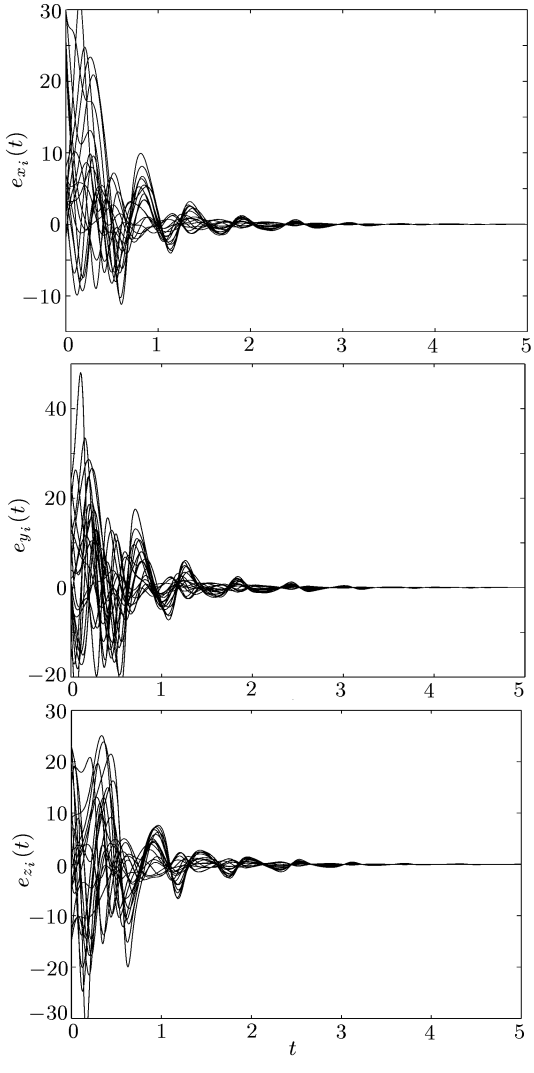


Fig. 2. Synchronous errors of the B–A network versus time t.

The Lorenz system has a chaotic attractor when

$\sigma = 10, \rho = 28$ and $b = 8/3$. Figure 2 illustrates the synchronous errors between node i and node $i+1$ of the above-mentioned B–A network: $e_{x_i(t)} = x_i(t) - x_{i+1}(t)$, $e_{y_i(t)} = y_i(t) - y_{i+1}(t)$ and $e_{z_i(t)} = z_i(t) - z_{i+1}(t)$, $i = 1, 2, \cdots, 19$, respectively. It is shown that the network of the B–A model with driving coupling (4) can easily come to synchronization. Analysis of stability of the synchronization will be given as follows.

Remark 1 We generate a B–A network of 20 nodes for illustrating that under the driving coupling, the network based on Lorenz systems can achieve synchronization. For convenience, we only generate 20 nodes such that Eq. (6) is written easily and the graph is clear, but this does not mean that a network with more nodes cannot achieve synchronization.

Since transversal Lyapunov exponents can help us to characterize the behaviour of infinitesimal vectors transversal to synchronized manifold, we calculate the maximal transverse Lyapunov exponent by applying the Master Stability function approach[9] to Eqs. (4). If the maximal transverse Lyapunov exponent is negative then the synchronized state is stable. However, we cannot reach a general relation between the Lyapunov exponents spectra and the eigenvalues of the coupled matrix, if the inner connected matrix is not a diagonal matrix in Eqs. (4). Therefore, we investigate the stability of synchronization in drivingly coupled network based Lorenz systems using numerical computation.

Consider an arbitrary network-based Lorenz system (4). Let $\lambda = \alpha + i\beta$ be the eigenvalue of coupled matrix, then the maximal transversal Lyapunov exponent mapping is a function of α and β. Let $L : \mathbb{C}^1 \to \mathbb{R}^1$ be a mapping on a complex plane, in which the maximal transversal Lyapunov exponents of networks are calculated by the master stability function approach. The stable region of synchronized state can be represented as

$$\Omega^s = \{x \in \mathbb{C}^1 \ | L(x) < 0\}, \tag{7}$$

the unstable region of synchronized state is

$$\Omega^u = \{x \in \mathbb{C}^1 \ | L(x) > 0\}, \tag{8}$$

and

$$\Omega^c = \{x \in \mathbb{C}^1 \ | L(x) = 0\} \tag{9}$$

is the boundary of Ω^s and Ω^u representing the maximal transverse Lyapunov exponent to be equal to zero, on which the points imply that the corresponding eigenvectors are along the direction of the synchronized manifold.

We plot a diagram of plane as shown in Fig. 3 which illustrates the regions of synchronized state. Note that point $(0, 0) \in \Omega^u$, implying that the maximal Lyapunov exponents is larger than zero, since this is just the case of isolated, chaotic Lorenz systems. If none of the eigenvalues of coupled matrix except for the zero eigenvalue falls into the unstable region Ω^u, namely, they are all the elements of $\Omega^s \cup \Omega^c$, then the synchronized state is stable. Correspondingly, if there is one eigenvalue of coupled matrix except for the zero eigenvalue which falls into the unstable region Ω^u, then the synchronized state is unstable.

From Fig. 3, on the real axis, the numbers in the stable region Ω^s are all less than -0.25. It can be easy to determine the stable synchronized state when all the eigenvalues of coupled matrix are real numbers. For the case of systems (4), the B–A network with linear driving coupling as shown in Fig. 1, we calculate the eigenvalues of coupled matrix as follows: (0.0000, -0.2942, -1.7477, -1.6958, -1.5835, -1.4986, -1.4725, -1.4003, -1.3477, -0.4356, -0.5019, -0.5665, -0.6253, -0.7474, -1.1980, -0.8480, -1.0625, -0.9744, -1.0000, -1.0000). The largest one of the above numbers except for zero is -0.2942 which is obviously less than -0.25. Then the synchronized state of B–A network with driving coupling (4) is stable and Fig. 2 shows the correctness.

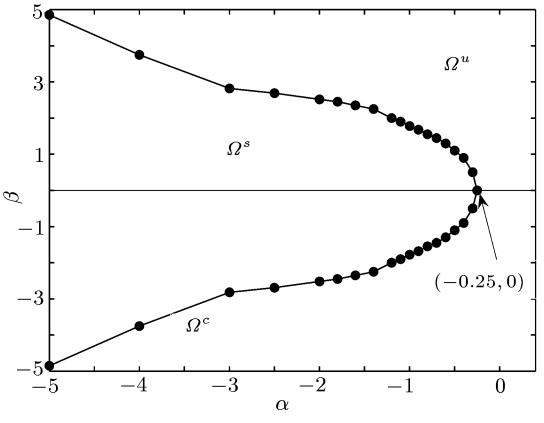


Fig. 3. Stability analysis of synchronous state.

Remark 2 Here the couple matrix of network (4) $\overline{G}$ is not a symmetric matrix, where $\overline{G} = KG$, $K = \mathrm{diag}\left(\frac{1}{k_1}, \ldots, \frac{1}{k_n}\right)$ is a diagonal matrix. For a given network, if the eigenvalues are all real numbers and the largest nonzero eigenvalues of the coupled matrix satisfies that $\lambda_{\max} \leq -\frac{d}{c}$, where d is a positive constant, c is the positive couple strength (in systems (4), $c = \rho$), then the network (4) can come to synchronization (see Refs. [3,4]). That is also why the region $(-0.25, 0)$ belongs to the unstable region.

Combining the method of calculating eigenvalue spectra with the numerical simulations of synchronous errors, a great deal of synchronous problems can be handled easily. Considering the cases of oscillators connected in all-to-all network, star-shape network,

ring-shape network (the nearest neighbour coupling) and chain-shape network, we obtain the following results.

(1) When oscillators connect in an all-to-all network based on Lorenz systems with linear driving coupling (4), the network will come to synchronization. Note that in the eigenvalues of the coupled matrix of all-to-all network, nonzero eigenvalues are equal to $-n/(n-1)$ which are in the stable region Ω^s, where n is the node number of the network.

(2) When oscillators connect in a star-shape network based on Lorenz systems with linear driving coupling (4), the network will come to synchronization. Note that in the eigenvalues of the coupled matrix of star-shape network, there are $n-2$ nonzero eigenvalues equal to -1, one nonzero eigenvalue is -2, which are all in the stable region Ω^s, where n is the node number of the network.

(3) When oscillators connect in a ring-shape network based on Lorenz systems with linear driving coupling (4), there are at most 8 oscillators that can come to synchronization. By calculating the eigenvalues of ring, we obtain the largest nonzero eigenvalues of the coupled matrix is equal to $-2\sin^2\left(\frac{\pi}{n}\right)$. When $n=8$, the largest nonzero eigenvalue $\lambda_{\max}=-0.2929$ belongs to the stable region Ω^s. However, when $n=9$, the largest nonzero eigenvalue $\lambda_{\max}=-0.2340$ is outside the stable region Ω^s.

(4) When oscillators connect in a chain-shape network based on Lorenz systems with linear driving coupling (4), there are at most 5 oscillators that can achieve synchronization. By calculating the eigenvalues of chain, we obtain the eigenvalues of the coupled matrix of 5 oscillators $(0,-0.2929,-1.0000,-2.0000,-1.7071)$ belonging to the stable region Ω^s except for zero eigenvalue. However, when $n=6$, the eigenvalues are $(0,-0.1910,-0.6910,-1.3090,-2.0000,-1.8090)$ in which the largest nonzero eigenvalue $\lambda_{\max}=-0.1910$ is in the unstable region Ω^u.

We omit plotting curves of synchronous errors here and it can be done easily by designing some sample programs.

Remark 3 Comparing the above results, one can conclude that the star-shape and all-to-all network can easily achieve synchronization and the synchronizabilty of chain-shape network is weaker than that of ring-shape, and they are all difficult to achieve synchronization when the number of the connected nodes is sufficient large.

In summary, we have further studied the synchronization of a new network model with driving coupling. The model of drivingly coupled network is the generalization of drive-response systems. Specifically, the stability of synchronized state in the network with linear driving coupling is investigated, the stable region and unstable region on the complex eigenvalue plane are established and the case of B–A network based on Lorenz system is analysed. Furthermore, the results of coupled in all-to-all, star-shape, ring-shape and chain-shape networks are obtained. Finally, the synchronous theory of the network with nonlinear driving coupling and the case of directed networks should be further investigated in the near future.

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