Noncontinuous Solitary Wave Solutions for Double sine-Gordon Equation *

TIAN Li-Xin(田立新)**, YU Shui-Meng(于水猛), NI Mei-Jie(倪美杰)

Nonlinear Science Research Center, Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, 212013

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Using the tanh method and a variable separated ordinary difference equation method to solve the double sine-Gordon equation, we derive some new exact travelling wave solutions, especially a new type of noncontinuous solitary wave solutions. These noncontinuous solitary wave solutions are verified by using the conservation law theory.

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The double sine-Gordon equation is significant because of its applications in many fields such as nonlinear optics, Josephson array, ferromagnetic material, charge density waves, and liquid helium, which offers simple and ideal models. Popov[1] presented a perturbation theory for the double sine-Gordon equation. Recently, more exact solutions of the double sine-Gordon equation have been obtained by using the F-expansion method.[2] The Jacobi-sn and Jacobi-cn function solutions to the double sine-Gordon equation were obtained by using the Jacobi elliptic function expansion with symbolic computation.[3] We previously studied exact solitary wave solutions of an approximate fully nonlinear double sine-Gordon equation by using the ansatz method.[4]

So far, many kinds of new solitons have been found. Rosenau and Hyman[5] studied the generalized nonlinear dispersive equation of $K(m,n)$ and obtained solitary wave solutions with compact support they called the compacton. In Ref. [6], we introduced a fifth-order $K(m,n)$ equation with nonlinear dispersion to obtain compacton solutions. Camassa and Holm[7] found peakon solutions, which have a discontinuous first-order derivative at the wave peaks. Yin and Tian[8] obtained peakon and compacton solutions For $K(p,q,1)$ equation. In Ref. [9], we also studied a generalized weakly dissipative CH equation and obtained peakon solutions. Zhang[10] studied the discontinuous solutions to the Euler equations for a van der Waals fluid, which contains one shock and one phase transition and have considered the general case that there is a characteristic between the shock front and the phase boundary. In this Letter, we obtain a new type of noncontinuous solutions, whether these noncontinuous solutions satisfy the conditions of shock wave solution or not will be reported elsewhere.

In this study, we investigate the double sine-Gordon equation

$$(u)_{tt} - (u)_{xx} + \alpha \sin(u) + \beta \sin[(2u)] = 0. \quad (1)$$

Firstly, we introduce the definition of the conservation law equation and proposition of noncontinuous solution according to Ref. [11].

Definition 1[11] If the equation set

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{k} A_i(u, x_1, \ldots, x_k, t)\frac{\partial u}{\partial x_i} = g(u, x_1, \ldots, x_k, t) \quad (2)$$

can be transformed to the form

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{k} \frac{\partial f_i(u, x_1, \ldots, x_k, t)}{\partial x_i} = g(u, x_1, \ldots, x_k, t), \quad (3)$$

then Eq. (2) is called the conservation law equation. If $g = 0$, Eq. (2) is called the homogeneous conservation law equation. Here u is an unknown vector function, $u = (u_1, \ldots, u_n)^T$. T denotes transposition and g is a known vector function, $g = (g_1, \ldots, g_n)^T$, A_i is a known matrix function ($i = 1, \ldots, k$), $A_i = \begin{pmatrix} a_{11}^{(i)} & \cdots & a_{1n}^{(i)} \\ \vdots & & \vdots \\ a_{n1}^{(i)} & \cdots & a_{nn}^{(i)} \end{pmatrix}$; x_i and t are called the spatial and time variables, respectively.

Now, we rewrite the double sine-Gordon equation in the form of conservation law. Let $u_x = v$, $u_t = w$. Thus $v_t = w_x = u_{xt}$. The double sine-Gordon equation can be rewritten as

$$\begin{cases} w_t - v_x + \alpha \sin u + \beta \sin(2u) = 0, \\ v_t = w_x, \\ u_t = w. \end{cases} \quad (4)$$

Namely, the conservation law of the double sine-Gordon equation reads

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}_t - \begin{bmatrix} 0 \\ w \\ v \end{bmatrix}_x = \begin{bmatrix} w \\ 0 \\ -\alpha \sin(u) - \beta \sin(2u) \end{bmatrix},$$

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** Email: tianlx@ujs.edu.cn

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where $f(u,x,t) = \begin{bmatrix} 0 \\ w \\ v \end{bmatrix}$, $g(u,x,t) = \begin{bmatrix} w \\ 0 \\ -\alpha\sin(u) - \beta\sin(2u) \end{bmatrix}$.

Definition 2[11] Let u be a weak solution of

$$\frac{\partial u}{\partial t} + \frac{\partial f(u,x,t)}{\partial x} = g(u,x,t). \tag{5}$$

Except for the first kind of discontinuity on finite curves, it preserves continuous one-order partial derivative in other definition region. Then u can be called a noncontinuous solution.

Proposition 1[11] The discontinuous vectors function $u(x,t)$ is a noncontinuous solution of Eq. (5), if and only if: (1) In the smooth parts, it is the classical solution of Eq. (5). (2) It satisfies $[u]s = [f]$ on discontinuous lines, where $\frac{dx}{dt} = s$ denotes the slope and $[u]$ denotes the jerk of u, $[u] = u(x+0,t) - u(x-0,t)$.

Now we study the exact solutions to the double sine-Gordon equation with the tanh method. For Eq. (1), let $\xi = x - ct$, in which c is the wave speed. Then Eq. (1) can be converted to

$$c^2(u)_{\xi\xi} - (u)_{\xi\xi} + \alpha\sin u + \beta\sin[(2u)] = 0. \tag{6}$$

Set $v = e^{iu}$, $i = \sqrt{-1}$. Then

$$u = \frac{1}{i}\ln v, \quad u' = \frac{v'}{iv}, \quad u'' = \frac{v''}{iv} - \frac{(v')^2}{iv^2}, \tag{7}$$

and we find

$$\sin(u) = \frac{v - v^{-1}}{2i}, \quad \sin(2u) = \frac{v^2 - v^{-2}}{2i}, \quad \cos(u) = \frac{v + v^{-1}}{2}. \tag{8}$$

This also gives

$$u = \arccos\left(\frac{v + v^{-1}}{2}\right). \tag{9}$$

Substituting transformations (7) and (8) into Eq. (6), we have the ordinary difference equation

$$\beta v^4 + \alpha v^3 - \alpha v - \beta + 2(c^2-1)vv'' - 2(c^2-1)(v')^2 = 0. \tag{10}$$

Balancing the highest order derivative term and the highest order nonlinear term of Eq. (10), which gives $m = 1$, we can set the solution in the form

$$v = a_0 + a_1 Y, \tag{11}$$

where a_0 and a_1 are unknown parameters, $Y = \tanh[\mu(x-ct)]$. Substitute Eq. (11) into Eq. (10), collect the coefficients of each power of Y and let the coefficients of Y^j $(j = 0, 1, \ldots)$ equal to zero. We can obtain a series of equations:

$$\begin{cases} \beta a_1^4 - 2a_1^2\mu^2 + 2a_1^2\mu^2c^2 = 0, \\ 4a_1\mu^2c^2a_0 - 4a_1\mu^2a_0 + 4\beta a_1^3a_0 + \alpha a_1^3 = 0, \\ 6\beta a_1^2a_0^2 + 3\alpha a_0 a_1^2 = 0, \\ 4a_1\mu^2a_0 - 4a_1\mu^2c^2a_0 + 4\beta a_0^3a_1 - \alpha a_1 = 0, \\ -\beta - 2a_1^2\mu^2c^2 - \alpha a_0 + 2a_1^2\mu^2 + \beta a_0^4 + \alpha a_1^3 = 0. \end{cases} \tag{12}$$

Using the symbolic computation programm MAPLE, we obtain

$$a_0 = -\frac{\alpha}{2\beta}, \quad a_1 = \pm\frac{\sqrt{\alpha^2 - 4\beta^2}}{2\beta}, \quad \mu = \pm\sqrt{\frac{\alpha^2 - 4\beta^2}{8(1-c^2)\beta}}, \tag{13}$$

where $\alpha > 2\beta$, $1 > c^2$, c is the wave speed. This gives

$$v(x,t) = -\frac{\alpha}{2\beta} \pm \frac{\sqrt{\alpha^2 - 4\beta^2}}{2\beta}\tanh[\mu(x-ct)], \tag{14}$$

$$v(x,t) = -\frac{\alpha}{2\beta} \pm \frac{\sqrt{\alpha^2 - 4\beta^2}}{2\beta}\coth[\mu(x-ct)]. \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (9), we obtain the exact solutions to Eq. (1) for $\alpha > 2\beta$, i.e.

$$u(x,t) = \arccos\left\{\frac{1}{2}\left(-\frac{\alpha}{2\beta} \pm \frac{\sqrt{\alpha^2-4\beta^2}}{2\beta}\tanh[\mu(x-ct)] + \frac{1}{-\frac{\alpha}{2\beta} \pm \frac{\sqrt{\alpha^2-4\beta^2}}{2\beta}\tanh[\mu(x-ct)]}\right)\right\}, \tag{16}$$

$$u(x,t) = \arccos\left\{\frac{1}{2}\left(-\frac{\alpha}{2\beta} \pm \frac{\sqrt{\alpha^2-4\beta^2}}{2\beta}\coth[\mu(x-ct)] + \frac{1}{-\frac{\alpha}{2\beta} \pm \frac{\sqrt{\alpha^2-4\beta^2}}{2\beta}\coth[\mu(x-ct)]}\right)\right\}. \tag{17}$$

On the other hand, for $\alpha < 2\beta$,

$$u(x,t) = \arccos\left\{\frac{1}{2}\left(-\frac{\alpha}{2\beta} \mp \frac{\sqrt{4\beta^2-\alpha^2}}{2\beta}\tan[\mu(x-ct)] + \frac{1}{-\frac{\alpha}{2\beta} \mp \frac{\sqrt{4\beta^2-\alpha^2}}{2\beta}\tan[\mu(x-ct)]}\right)\right\}, \tag{18}$$

$$u(x,t) = \arccos\left\{\frac{1}{2}\left(-\frac{\alpha}{2\beta} \mp \frac{\sqrt{4\beta^2-\alpha^2}}{2\beta}\cot[\mu(x-ct)] + \frac{1}{-\frac{\alpha}{2\beta} \mp \frac{\sqrt{4\beta^2-\alpha^2}}{2\beta}\cot[\mu(x-ct)]}\right)\right\}, \tag{19}$$

where $\mu = \pm\sqrt{\frac{\alpha^2 - 4\beta^2}{8(1-c^2)\beta}}$.

In what follows, we employ a variable separated ordinary difference equation method to formally derive a new type of travelling wave solution to Eq. (1) by using the variable $\xi = x - ct$, which gives

$$u'' + \frac{\alpha}{c^2-1}\sin(u) + \frac{\beta}{c^2-1}\sin[(2u)] = 0, \tag{20}$$

where the prime denotes the derivative with respect to ξ, $c^2 \neq 1$. We assume that $u(\xi)$ satisfies a variable separated ordinary difference equation given by

$$u'(\xi) = \frac{du}{d\xi} = a + b\cos u, \tag{21}$$

where a and b are the parameters determined in the following. Differentiating Eq. (21) with respect to ξ yields

$$u''(\xi) + ab\sin u + \frac{b^2}{2}\sin(2u) = 0. \tag{22}$$

Comparing Eq. (20) with Eq. (22), we obtain

$$ab = \frac{\alpha}{c^2-1}, \quad \frac{b^2}{2} = \frac{\beta}{c^2-1}. \tag{23}$$

Then

$$a = \frac{\alpha}{\sqrt{2\beta(c^2-1)}}, b = \sqrt{\frac{2\beta}{c^2-1}}. \tag{24}$$

We set Eq. (21) as follows:

$$\frac{1}{a+b\cos(u)}du = d\xi. \tag{25}$$

Integrating both the sides of Eq. (25), we can find the solutions

$$\frac{2}{\sqrt{a^2-b^2}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{u}{2}\right)\right) = \xi + \xi_0, \tag{26}$$

$$\frac{2}{\sqrt{b^2-a^2}}\arctan h\left(\sqrt{\frac{b-a}{b+a}}\tan\left(\frac{u}{2}\right)\right) = \xi + \xi_0, \tag{27}$$

$$\frac{2}{\sqrt{b^2-a^2}}arc\coth\left(\sqrt{\frac{b-a}{b+a}}\tan\left(\frac{u}{2}\right)\right) = \xi + \xi_0, \tag{28}$$

where ξ_0 is the integral constant. Then we can give the exact solutions

$$u(x,t) = 2\arctan\left\{\sqrt{\frac{\alpha+2\beta}{\alpha-2\beta}} \cdot \tan\left(\sqrt{\frac{\alpha^2-4\beta^2}{8\beta(c^2-1)}}(x-ct)+\xi_0\right)\right\}, \quad \alpha > 2\beta, \tag{29}$$

which is a periodic solitary wave solution as shown by curve 1 in Fig. 1 with $\alpha = 4$, $\beta = 1$, $c = 2$.

$$u(x,t) = -2\arctan\left\{\sqrt{\frac{\alpha+2\beta}{2\beta-\alpha}} \cdot \tanh\left(\sqrt{\frac{4\beta^2-\alpha^2}{8\beta(c^2-1)}}(x-ct)+\xi_0\right)\right\}, \quad \alpha < 2\beta, \tag{30}$$

which is a kink solution as shown by curve 2 in Fig. 1 with $\alpha = 1$, $\beta = 2$, $c = 2$, $\xi_0 = 0$.

$$u(x,t) = 2\arctan\left\{\sqrt{\frac{2\beta-\alpha}{\alpha+2\beta}} \cdot \tanh\left(\sqrt{\frac{4\beta^2-\alpha^2}{8\beta(c^2-1)}}(x-ct)+\xi_0\right)\right\}, \quad \alpha < 2\beta, \tag{31}$$

which is an anti-kink solution as shown by curve 3 in Fig. 1 with $\alpha = 1$, $\beta = 2$, $c = 2$, $\xi_0 = 0$.

$$u(x,t) = 2\arctan\left\{\sqrt{\frac{2\beta-\alpha}{\alpha+2\beta}} \cdot \coth\left(\sqrt{\frac{4\beta^2-\alpha^2}{8\beta(c^2-1)}}(x-ct)+\xi_0\right)\right\}, \quad \alpha < 2\beta, \tag{32}$$

which is a noncontinuous solution as shown by curve 4 in Fig. 1 with $\alpha = 1$, $\beta = 2$, $c = 2$, $\xi_0 = 0$.

When $a = b$, Eq. (25) becomes

$$\frac{1}{1+\cos(u)}du = ad\xi. \tag{33}$$

By integrating we obtain

$$\tan\left(\frac{u}{2}\right) = a(\xi + \xi_0). \tag{34}$$

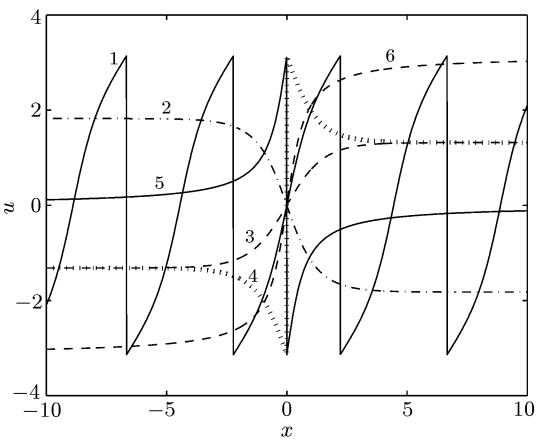


Fig. 1. Solitary wave solution and noncontinuous solution of double sine-Gordon equation: (1) Periodic solitary wave solution with $\alpha = 4$, $\beta = 1$, $c = 2$; (2) kink solution with $\alpha = 1$, $\beta = 2$, $c = 2$, $\xi_0 = 0$; (3) anti-kink solution with $\alpha = 1$, $\beta = 2$, $c = 2$, $\xi_0 = 0$; (4) noncontinuous solution with $\alpha = 1$, $\beta = 2$, $c = 2$, $\xi_0 = 0$; (5) anti-kink solution with $c = 2$, $\alpha = 3$, $\xi_0 = 0$; (6) noncontinuous solution with $c = 2$, $\alpha = 3$, $\xi_0 = 0$.

Therefore we find

$$u(x,t) = 2\arctan\left(\frac{\alpha}{\sqrt{c^2-1}}[(x-ct)+\xi_0]\right), \tag{35}$$

which is an anti-kink solution as shown curve 5 in Fig. 1 with $c = 2$, $\alpha = 3$, $\xi_0 = 0$.

Considering the other case, i.e. $a = -b$, the equation becomes

$$\frac{1}{1-\cos(u)}du = ad\xi. \tag{36}$$

We find the solution

$$u(x,t) = 2\mathrm{arc}\cot\left(-\frac{\alpha}{\sqrt{c^2-1}}[(x-ct)+\xi_0]\right), \tag{37}$$

which is a noncontinuous solution as shown by curve 6 in Fig. 1 with $c = 2$, $\alpha = 3$, $\xi_0 = 0$.

Then we prove that the noncontinuous solutions (32) and (37) of the double sine-Gordon equation are noncontinuous solitary wave solutions.

Corollary 1 Solution (32) of the double sine-Gordon equation

$$u(x,t) = 2\arctan\left\{\sqrt{\frac{2\beta-\alpha}{\alpha+2\beta}} \cdot \coth\left(\sqrt{\frac{4\beta^2-\alpha^2}{8\beta(c^2-1)}}(x-ct)+\xi_0\right)\right\} \quad \text{for } \alpha < 2\beta$$

is a noncontinuous solution.

Proof It is easily known that (1) Eq. (32) is the classical solution of the double sine-Gordon equation in the smooth parts and (2) the discontinuous line is $\xi = -\sqrt{\frac{8\beta(c^2-1)}{4\beta^2-\alpha^2}}\xi_0$.

We will prove that the expression $[u^*]s = [f]$ holds on discontinuous lines. Here $u^* \equiv \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, $s = \frac{dx}{dt} = c$, $f = \begin{bmatrix} 0 \\ w \\ v \end{bmatrix}$.

(i) $[u] = u(0^+,t) - u(0^-,t) = 0$.

(ii) $v = u_x = u_\xi = 2\sqrt{\frac{2\beta-\alpha}{\alpha+2\beta}}\sqrt{\frac{4\beta^2-\alpha^2}{8\beta(c^2-1)}}\left\{1 - \coth^2\left[\sqrt{\frac{4\beta^2-\alpha^2}{8\beta(c^2-1)}}(x-ct)+\xi_0\right]\right\}\left\{1+\left\{\sqrt{\frac{2\beta-\alpha}{\alpha+2\beta}}\coth\left[\sqrt{\frac{4\beta^2-\alpha^2}{8\beta(c^2-1)}}(x-ct)+\xi_0\right]\right\}^{-2}\right.$, thus [v] = 0.

(iii) $w = u_t = -cu_\xi = -2c\sqrt{\frac{2\beta-\alpha}{\alpha+2\beta}}\sqrt{\frac{4\beta^2-\alpha^2}{8\beta(c^2-1)}}$ $\left\{1 - \coth^2\left[\sqrt{\frac{4\beta^2-\alpha^2}{8\beta(c^2-1)}}(x-ct)+\xi_0\right]\right\}\left\{1+\left\{\sqrt{\frac{2\beta-\alpha}{\alpha+2\beta}}\right.\right.$ $\left.\left.\coth\left[\sqrt{\frac{4\beta^2-\alpha^2}{8\beta(c^2-1)}}(x-ct)+\xi_0\right]\right\}^{-2}\right.$, thus [w] = 0.

Therefore, $[u^*] = 0$, $[f] = 0$. Namely, case (2) holds. We complete the proof of the corollary.

Corollary 2 Solution (37) of the double sine-Gordon equation

$$u(x,t) = 2\mathrm{arc\,cot}\left\{-\frac{\alpha}{\sqrt{c^2-1}}\coth\left[(x-ct)+\xi_0\right]\right\}$$

is a noncontinuous solution.

Proof It is easily known that (1) Eq. (37) is the classical solution of the double sine-Gordon equation in the smooth parts and (2) the discontinuous line is $\xi = -\xi_0$, where $\xi = x - ct$.

We will prove that the expression $[u^*]s = [f]$ holds on discontinuous lines.

Here $u^* \equiv \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, $f = \begin{bmatrix} 0 \\ w \\ v \end{bmatrix}$.

(i) $[u] = u(0^+,t) - u(0^-,t) = 0$.

(ii) $v = u_x = u_\xi$

$$= \frac{\frac{2\alpha}{\sqrt{c^2-1}}}{1+\left\{-\frac{\alpha}{\sqrt{c^2-1}}[(x-ct)+\xi_0]\right\}^2}, \quad \text{thus}$$

[v] = 0.

(iii) $w = u_t = -cu_\xi$

$$= \frac{\frac{-2c\alpha}{\sqrt{c^2-1}}}{1+\left\{-\frac{\alpha}{\sqrt{c^2-1}}[(x-ct)+\xi_0]\right\}^2}, \quad \text{thus}$$

[w] = 0.

Therefore, $[u^*] = 0$, $[f] = 0$. Namely, case (2) holds. This completes the proof of the corollary.

In summary, we have derived abundant exact solutions by the tanh method and a variable separated ordinary difference equation method.

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