Quantum Stackelberg Duopoly of Continuous Distributed Asymmetric Information

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(Received 24 June 2007)

The minimal quantization structure is employed to investigate the quantum version of the Stackelberg duopoly with continuous distributed asymmetric information, i.e. the first mover has incomplete information that obeys a continuous distribution while the second mover has complete information. It is found that the effects of the positive quantum entanglement on the outcomes exhibit many interesting features due to the information asymmetry. Moreover, although the first-mover advantage is counteracted by the information asymmetry, the positive quantum entanglement still enhances the first-mover advantage and improves the first-mover tolerance of the information asymmetry beyond the classical limit.

PACS: 03. 67. − a, 02. 50. Le

The game theory has experienced a great development since the early last century through its wide application in modelling many nature and social phenomena. In recent years, the concept of quantum information was applied to the classical game theory and the quantum game theory has emerged as a new active field.[1] This field was initialized by Meyer, in which a player can always beat his classical opponent by adopting a quantum strategy in the coin tossing game.[2] Next, Eisert *et al.* presented the quantization of the famous prisoners' dilemma and eliminated the dilemma existing in the classical game by quantum strategies.[1] Since then, many interesting and surprising properties of the quantum game have been investigated theoretically.[3−11] Furthermore, the first experimental realization of the quantum game has been successively accomplished,[12] which constructs a practical base for the quantum game.

In 2002, Li *et al.*[3] investigated the quantization of the game with continuous strategic space instead of discrete strategies, and gave a 'minimal' quantum structure of the Cournot duopoly. The results showed that the dilemma-like situation can disappear by quantizing the game. These interesting features are completely attributed to the quantum entanglement due to the minimal quantum structure. Since then, the Cournot and Bertrand duopolies in different cases have been quantized and investigated with the minimal quantum structure.[4−7] Many kinds of interesting and surprising properties have been obtained. A multiplayer version of the minimal quantum structure was also proposed.[8]

As we know, the Cournot and Bertrand duopolies both are static games, in which all players make their strategic move at the same time and must guess the actions of their opponents. Apart from the static games, there is another distinct kind of games, i.e. dynamic games, where all players make their strategic move at different time. The Stackelberg duopoly is commonly regarded as the simplest dynamic game,[9] in which one of the two firms (firm 1) is allowed to move first. The competitor (firm 2) can observe moving of the opponent before making its own decision. In such a dynamic game there exists an interesting result, the first-mover advantage. A natural question is what role the quantum entanglement will play in the quantum Stackelberg duopoly. In 2003, Lo *et al.*[10] applied the minimal quantum structure to the Stackelberg duopoly with complete information, and obtained that positive quantum entanglement would enhance the first-mover advantage beyond the classical limit. They also studied the Stackelberg duopoly with discrete distributed asymmetric information,[11] and obtained the similar results. Moreover, the first-mover advantage is eradicated due to the first-mover asymmetric information though positive quantum entanglement improves the first-mover tolerance of the asymmetric information.

Apart from the games with discrete distributed information, the games with continuous distributed information, which have much realistic significance in practical markets, should be of particular interest.[13] In this Letter, we apply the minimal quantization structure of Li *et al.* to the quantum version of the Stackelberg duopoly with continuous distributed asymmetric information. It is assumed that the first mover has incomplete information that obeys a specific continuous distribution, while the second mover has complete information. Following the recent study in Ref. [11], we evaluate the effects of the positive

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quantum entanglement and the information asymmetry on the first-mover advantage. Moreover, we also evaluate the effects of the positive quantum entanglement and the information asymmetry on the profit outcomes of the two firms. These results show that the effects of the positive quantum entanglement on the outcomes exhibit many interesting features due to the information asymmetry. Moreover, although the first-mover advantage is counteracted by the information asymmetry, the positive quantum entanglement still enhances the first-mover advantage and improves the first-mover tolerance of the information asymmetry beyond the classical limit. We can find that the main conclusions for the continuous distribution are the same as those for the discrete distribution in Ref. [11].

We first review the classical Stackelberg duopoly with continuous distributed asymmetric information, in which the first and second movers, i.e. firms 1 and 2, produce a homogeneous product for the market. Firm 2 clearly knows the firm-1 unit cost c_1 and firm 1 only knows the probability distribution $f_2(c_2)$ of the firm-2 unit cost c_2. Accordingly, profit functions of the two firms can be described as

$$\begin{aligned}\mu_1(q_1,q_2) &= \int f_2(c_2)q_1(a-c_1-q_1-q_2)dc_2,\\ \mu_2(q_1,q_2) &= q_2(a-c_2-q_1-q_2),\end{aligned} \tag{1}$$

where q_1 and q_2 are the product quantities of firms 1 and 2, respectively.

The solution to this duopoly problem can be obtained by the methods of subgame perfect equilibrium and backwards induction. This means that we first solve Eq. (1) for firm 2, i.e. firm 2 reacts optimally to q_1 by choosing q_2 which maximizes its profit. Then firm 1 takes into account the optimal response of firm 2 and sets its own moving in such a way to maximize its own profit. That is, firm 1 substitutes the firm-2 reaction function into its own profit function μ_1, and then optimizes it by setting the first derivative of μ_1 with respect to q_1 equal to zero. Consequently, the quantities of two firms can be obtained to read

$$\begin{aligned}q_1^* &= \frac{1}{2}(a-c_1),\\ q_2^* &= \frac{1}{4}(a+c_1-2c_2),\end{aligned} \tag{2}$$

where we have assumed that $\bar{c}_2 = c_1$ for simplicity. The strategic choices $\{q_1^*, q_2^*\}$ constitute the Bayes–Nash equilibrium. The corresponding profit functions are given by

$$\begin{aligned}\overline{\mu_1^*}(q_1^*,q_2^*) &= \frac{1}{8}k^2,\\ \overline{\mu_2^*}(q_1^*,q_2^*) &= \frac{1}{4}k^2\Big(\frac{1}{4}+\xi\Big),\end{aligned} \tag{3}$$

where $k = a - c_1$, and the parameter

$$\xi \equiv \frac{\overline{\Delta c_2^2}}{k^2} \tag{4}$$

represents a measure of the information asymmetry, $\overline{\Delta c_2^2} = \int f_2(c_2)(c_2-\bar{c_2})^2 dc_2$ is the information fluctuation of the firm-2 unit cost. As a result, the difference between the two-firm expected profit functions can be evaluated as

$$\delta\bar{\mu}^*_{\rm classical} = \bar{\mu}_1^* - \bar{\mu}_2^* = \frac{1}{4}k^2\Big(\frac{1}{4}-\xi\Big). \tag{5}$$

It is clear that there exists a critical value of ξ, namely $\xi_c^{\rm classical} = 1/4$, beyond which firm 1 will lose its advantage. In other words, asymmetric information of firm 1 will eradicate its first-mover advantage. Obviously, the standard solution to the classical Stackelberg duopoly of complete information is recovered with $\xi = 0$.

In the following, we apply the minimal quantization structure of Li *et al.* to the quantum extension of the Stackelberg duopoly with continuous distributed asymmetric information. In the quantization scheme,[3] the game starts from the state $|{\rm vac}\rangle_1 \otimes |{\rm vac}\rangle_2$, which is the direct product of two single-mode electromagnetic fields. This state undergoes a unitary entanglement operation $\hat{J}(\gamma) \equiv \exp\{-\gamma(\hat{a}_1^+\hat{a}_2^+ - \hat{a}_1\hat{a}_2)\}$. It is known to both firms and symmetric with respect to interchange of the two field modes. Here $\gamma \geq 0$ is known as the squeezing parameter and can be regarded as a measure of the quantum entanglement, the operators $\hat{a}_1(\hat{a}^+)$ and $\hat{a}_2(\hat{a}_2^+)$ are the annihilation (creation) operators of the electromagnetic fields. Strategic movers of two firms are associated with the unitary operations $\hat{D}_1(x_1) \equiv \exp\{x_1(\hat{a}_1^+ - \hat{a}_1)/\sqrt{2}\}$ and $\hat{D}_2(x_2) \equiv \exp\{x_2(\hat{a}_2^+ - \hat{a}_2)/\sqrt{2}\}$, respectively. Having executed their moves, both the firms forward their electromagnetic field modes to the final measurement, prior to which a disentanglement operation $\hat{J}^+(\gamma)$ is carried out. Therefore the final state prior to the measurement is equal to

$$|\psi_f\rangle \equiv \hat{J}(\gamma)^+\hat{D}_1(x_1)\hat{D}_2(x_2)\hat{J}(\gamma)|{\rm vac}\rangle_1 \otimes |{\rm vac}\rangle_2, \tag{6}$$

which is a product of two coherent states $|q_1\rangle_1$ and $|q_2\rangle_2$. In order to reduce the noise of the measurement result, the final measurement is carried out by homodyne measurement with an infinitely squeezed reference light beam.[3] The measured results are the eigenvalues of the final coherent states and represent the quantities of two firms denoted by

$$\begin{aligned}q_1 &= x_1\cosh\gamma + x_2\sinh\gamma,\\ q_2 &= x_1\sinh\gamma + x_2\cosh\gamma.\end{aligned} \tag{7}$$

Substituting Eq. (7) into Eq. (1), profit functions of

the two firms are given by

$$\begin{aligned}\mu_1(q_1,q_2) =& \int f_2(c_2)(x_1\cosh\gamma + x_2\sinh\gamma)\\ &\cdot(a-c_1-x_1e^{\gamma}-x_2e^{\gamma})dc_2,\\ \mu_2(q_1,q_2) =& (x_1\sinh\gamma + x_2\cosh\gamma)\\ &\cdot(a-c_2-x_1e^{\gamma}-x_2e^{\gamma}).\end{aligned} \tag{8}$$

Similar to the classical case, the solution to this duopoly problem is easy to work out by the methods of subgame perfect equilibrium and backwards induction. The quantities of the two firms are evaluated to be

$$\begin{aligned}q_1^* &= k\frac{e^{-\gamma}\cosh^2\gamma}{1+e^{-\gamma}\cosh\gamma},\\ q_2^* &= \frac{1}{2}e^{-\gamma}\Big(a-c_2-\frac{e^{\gamma}\cosh\gamma}{1+e^{-\gamma}\cosh\gamma}k\Big).\end{aligned} \tag{9}$$

The corresponding expected profit functions are expressed as

$$\begin{aligned}\overline{\mu_1^*}(q_1^*,q_2^*) =& \frac{1}{4}k^2\Big[-e^{-\gamma\sinh\gamma}\xi\\ &+\frac{1}{1+e^{-\gamma}\cosh\gamma}\Big],\\ \overline{\mu_2^*}(q_1^*,q_2^*) =& \frac{1}{4}k^2\Big[e^{-\gamma}\cosh\gamma\xi\\ &+\frac{\cosh\gamma e^{-\gamma}}{(1+e^{-\gamma}\cosh\gamma)^2}\Big].\end{aligned} \tag{10}$$

It is easy to see that these forms revert to those of the discrete distributed asymmetric information in Ref. [11] when we take the special values of

$$f_2(c_2) = \begin{cases}\theta, & c_2 = c_{2H},\\ 1-\theta, & c_2 = c_{2L}.\end{cases}$$

Next we investigate the influence of the variables ξ and γ. In the expression of the firm-1 expected profit function $\overline{\mu_1^*}$, $-\sinh\gamma e^{-\gamma}\xi$ gradually decreases with increasing γ for fixed ξ, while $\dfrac{1}{1+e^{-\gamma}\cosh\gamma}$ gradually increases with increasing γ. Therefore the dependence of $\overline{\mu_1^*}$ on γ is dramatically changed due to the information asymmetry ξ. The derivative of $\overline{\mu_1^*}$ with respect to γ is given by

$$\frac{\partial\overline{\mu_1^*}}{\partial\gamma} = \frac{1}{4}k^2e^{-2\gamma}\Big(\frac{1}{(1+e^{-\gamma}\cosh\gamma)^2}-\xi\Big). \tag{11}$$

It can be obtained from the above equation that there are two thresholds for the information asymmetry ξ, i.e. $\xi_{\rm th1} = 1/4$ and $\xi_{\rm th2} = 4/9$. When $\xi < \xi_{\rm th1}$, $\dfrac{\partial\overline{\mu_1^*}}{\partial\gamma} < 0$ for any $\gamma \geq 0$ and $\overline{\mu_1^*}$ decreases monotonically with increasing γ. When $\xi > \xi_{\rm th2}$, $\dfrac{\partial\overline{\mu_1^*}}{\partial\gamma} > 0$ for any $\gamma \geq 0$, and $\overline{\mu_1^*}$ increases monotonically with increasing γ. When $\xi_{\rm th1} < \xi < \xi_{\rm th2}$, there is γ_m such that $\dfrac{\partial\overline{\mu_1^*}}{\partial\gamma} > 0$ for $\gamma > \gamma_m$ and $\dfrac{\partial\overline{\mu_1^*}}{\partial\gamma} < 0$ for $\gamma < \gamma_m$. It follows that $\overline{\mu_1^*}$ increases monotonically with increasing γ for $\gamma > \gamma_m$ but decreases monotonically with increasing γ for $\gamma < \gamma_m$. On the other hand, $\overline{\mu_1^*}$ always decreases with increasing ξ for fixed γ, this means that firm-1 asymmetric information will reduce its expected profit. Firm-1 expected profit $\overline{\mu_1^*}$ as a function of ξ and γ is given in Fig. 1. For the sake of clarity, several cross sections of Fig. 1 are shown in Fig. 2, in which all features can be seen.

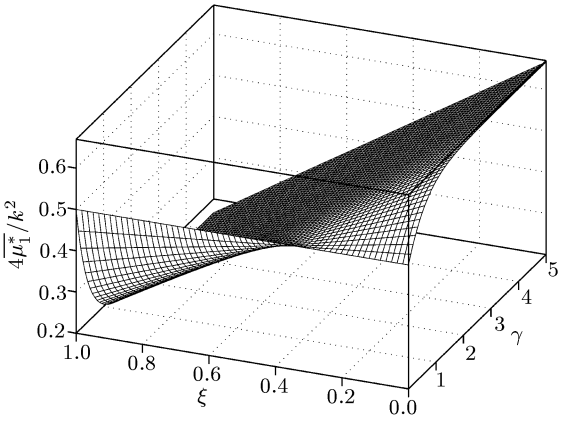


Fig. 1. Profit function $\overline{\mu_1^*}$ versus the quantum entanglement γ and the information asymmetry ξ.

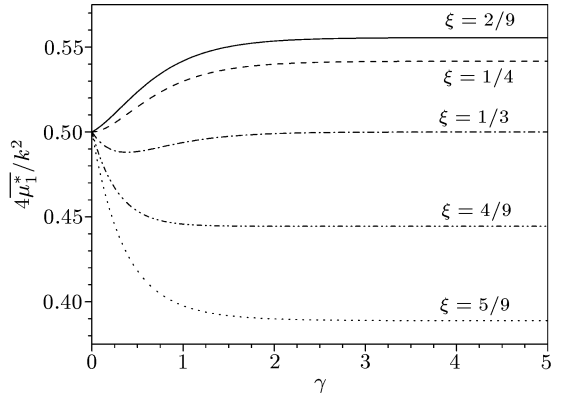


Fig. 2. Profit function $\overline{\mu_1^*}$ versus the quantum entanglement γ with different setting of ξ.

From Eq. (10), it follows that the firm-2 expected profit function $\overline{\mu_2^*}$ always decreases with increasing γ for fixed ξ. On the other hand, $\overline{\mu_2^*}$ always increases with increasing ξ for fixed γ. This means that the positive quantum entanglement γ will reduce the firm-2 expected profit, but firm-1 information asymmetry will enhance firm-2 expected profit. The firm-2 expected profit $\overline{\mu_2^*}$ is plotted as functions of ξ and γ in Fig. 3, where all the features can also be seen.

As a result, the difference between the expected profits of the two firms can be evaluated as

$$\begin{aligned}\delta^*_{\rm quantum} &= \overline{\mu_1^*}(q_1^*, q_2^*) - \overline{\mu_2^*}(q_1^*, q_2^*) \\ &= \frac{1}{4}k^2\Big[\frac{1}{(1+e^{-\gamma}\cosh\gamma)^2} - \xi\Big].\end{aligned} \tag{12}$$

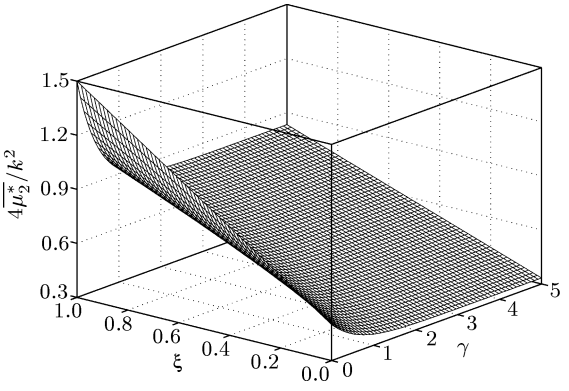


Fig. 3. Profit function $\overline{\mu_2^*}$ versus the quantum entanglement γ and the information asymmetry ξ.

It is obvious that the profit difference $\delta^*_{\rm quantum}$ decreases with increasing ξ, which denotes that the firm-1 information asymmetry will counteract the first-mover advantage. On the other hand, the profit difference $\delta^*_{\rm quantum}$ grows monotonically with increasing γ. Hence the first-mover advantage is enhanced by the positive quantum entanglement. The profit difference $\delta^*_{\rm quantum}$ against ξ and γ is plotted in Fig. 4.

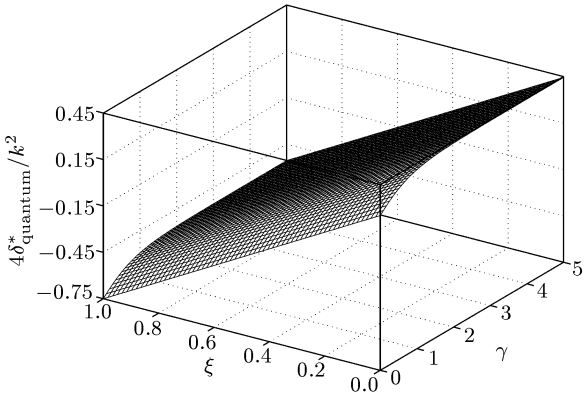


Fig. 4. Profit difference $\delta^*_{\rm quantum}$ versus the quantum entanglement γ and the information asymmetry ξ.

As shown in Eq. (12), firm 1 maintains the first-mover advantage provided that ξ is less than the critical value

$$\xi_c^{\rm quantum} = \frac{1}{[1+e^{-\gamma}\cosh\gamma]^2}, \tag{13}$$

which denotes that the first-mover tolerance of the information asymmetry grows monotonically with γ, starting from the minimum value $1/4$ with $\gamma = 0$ and reaching the maximum $4/9$ with the limit of $\gamma \to \infty$. In other words, the first-mover tolerance of the information asymmetry is enhanced beyond the classical limit by the positive quantum entanglement.

In summary, we apply the minimal quantization structure of Li *et al.* to the quantum version of Stackelberg duopoly, in which the first mover has incomplete information that obeys a continuous distribution while the second mover has complete information. In contrast to the case of complete information where the first-mover outcome increases with the positive quantum entanglement, the effects of the quantum entanglement on the outcomes of the two players are dramatically changed due to the information asymmetry. Furthermore, although the first-mover advantage is counteracted by the first-mover information asymmetry, the positive quantum entanglement still enhances the first-mover advantage and improves first-mover tolerance of information asymmetry in the classical limit.

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