Cooper-Pair Number-Phase Quantization for a Mesoscopic LC Circuit Including a Josephson Junction *

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By introducing the entangled state representation and Feynman assumption that 'electron pairs are bosons, ..., a bound pair acts as a Bose particle', we construct an operator Hamiltonian for a mesoscopic inductance-capacitance (LC) circuit including a Josephson junction, then we use the Heisenberg equation of motion to derive the current equation and the voltage equation across the inductance as well as across the Josephson junction. The result manifestly shows how the junction voltage is affected by the capacitance coupling. In this way the Cooper-pair number-phase quantization for this system is completed.

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In recent years, with the progress of the quantum computation and quantum information, it is recognized that solid state quantum computation is promising for the quantum circuit integration, thereby a practical quantum computer can be built. The Josephson junction,[1−3] which is comprised of two superconductors 'weakly' connected by a thin layer of insulating material, would play an important role in realizing the solid quantum computation.[4,5]

For designing practical quantum computers, in addition to Josephson junctions, the mesoscopic circuit including the elementary components such as capacitance, inductance and electric resistance, is also required. In this Letter, we propose an LC mesoscopic circuit including a Josephson junction, i.e. a Josephson junction is coupled to an inductance by a capacitor. The circuit is excited by an instant impulse source. We investigate how the Josephson equations change due to the existence of coupling. For this aim, following Feynman's assumption[6] that "electron pairs are bosons, ..., nearly all of the pairs will be locked down at the lowest energy in exactly the same state', we propose a bosonic operator model for describing the mesoscopic circuit including a Josephson junction, i.e. construct a quantum mechanical Hamiltonian (a quantization scheme), then we use the entangled state representation to propose Cooper-pair number-phase quantization. Furthermore, we use the Heisenberg equation to derive the current equation and the voltage equation across the Josephson junction. The result manifestly shows the role of the capacitance coupling between the Josephson junction and the inductance. Following Louiselle's quantization of a simple LC circuit,[7] we quantize the circuit in Fig. 1 by the Cooper-pair number-phase quantization scheme. Before doing this, we analyse this circuit from the point of view of classical Hamilton dynamics.

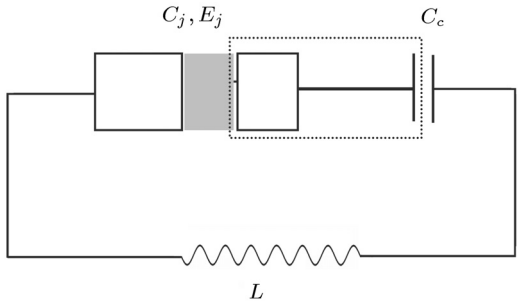


Fig. 1. The mesoscopic LC circuit including a Josephson junction.

A single Josephson junction is characterized by two Josephson equations (current equation and junction voltage equation)

$$I_j = I_c \sin\varphi_j, \tag{1}$$

$$\dot{\varphi}_j = \frac{2eu_j}{\hbar}, \tag{2}$$

where the subscript j represents the Josephson junction, $I_c = 2eE_j/\hbar$ is the critical electric current of the Josephson junction, u_j is the voltage drop across the Josephson junction, and φ_j is the phase difference between two superconductors of a junction. Thus, when tunnelling occurs, the work performed across the junction is

$$\int_0^t u_j I_c \sin\varphi_j dt = E_j(\cos\varphi_j - 1). \tag{3}$$

For the non-dissipative inductance, from Faraday's law of electromagnetic induction we can obtain the

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voltage drop u_L across the inductance

$$u_L = \dot{\Phi}_L, \tag{4}$$

where Φ_L is the self-inductance magnetic flux through the inductance.

We employ the Hamilton dynamics to analyse the circuit by defining φ_j and Φ_L as generalized coordinates, the system potential energy is

$$\mathcal{V} = \frac{1}{2L}\Phi_L^2 + E_j(1-\cos\varphi_j). \tag{5}$$

On the other hand, the charging energy in all the capacitors is

$$\mathcal{T} = \frac{1}{2}C_c u_c^2 + \frac{1}{2}C_j u_j^2, \tag{6}$$

where C_c is the coupling capacitance, and $\mathcal{T}$ is considered as the kinetic energy, since from Eqs. (2) and (4) one can see u_j and u_c are, respectively, related to the generalized velocity $\dot{\Phi}_L$ and $\dot{\varphi}_j$. From the simple analysis of voltage at the nodes of the circuit we can obtain

$$u_c = u_j + u_L. \tag{7}$$

Substituting Eqs. (2), (4) and (7) into Eq. (6) yields

$$\mathcal{T} = \frac{1}{2}(C_c+C_j)\Big(\frac{\hbar}{2e}\Big)^2\dot{\varphi}_j^2 + \frac{1}{2}C_c\dot{\Phi}_L^2 + \frac{\hbar}{2e}C_c\dot{\varphi}_j\dot{\Phi}_L. \tag{8}$$

Then the Lagrangian function of the system is

$$\begin{aligned}\mathcal{L} &= \mathcal{T}-\mathcal{V}\\ &= \frac{1}{2}(C_c+C_j)\Big(\frac{\hbar}{2e}\Big)^2\dot{\varphi}_j^2 + \frac{1}{2}C_c\dot{\Phi}_L^2 + \frac{\hbar}{2e}C_c\dot{\varphi}_j\dot{\Phi}_L\\ &\quad - \frac{1}{2L}\Phi_L^2 - E_j(1-\cos\varphi_j).\end{aligned} \tag{9}$$

From the charge neutrality law (or electric current continuity) at nodes of the circuit, $u_jC_j - 2ne = -u_cC_c$, then from Eqs. (2), (4) and (7), we have

$$2ne = u_jC_j + u_cC_c = (C_j+C_c)\frac{\hbar}{2e}\dot{\varphi}_j + C_c\dot{\Phi}_L, \tag{10}$$

where n is the excess number of Cooper-pairs on the island. Using Eqs. (2), (4), (9) and (10) we calculate the generalized momenta, respectively, conjugated to φ_j and Φ_L

$$p_j \equiv \frac{\partial\mathcal{L}}{\partial\dot{\varphi}_j} = \frac{\hbar}{2e}\Big[(C_j+C_c)\frac{\hbar}{2e}\dot{\varphi}_j + C_c\dot{\Phi}_L\Big] = n\hbar, \tag{11}$$

$$p_L \equiv \frac{\partial\mathcal{L}}{\partial\dot{\Phi}_L} = C_c\dot{\Phi}_L + \frac{\hbar}{2e}C_c\dot{\varphi}_j. \tag{12}$$

Note that the generalized momenta p_j is proportional to the excess number n of Cooper-pairs, which implies the possibility of number-phase quantization, as we will carry out in the following. From Legendre transformation and Eqs. (11) and (12) we can obtain the Hamiltonian

$$H = p_j\dot{\varphi}_j + p_L\dot{\Phi}_L - \mathcal{L} \equiv H_j + H_L + H_{\rm int}, \tag{13}$$

where

$$H_j = E_c^{(j)}n^2 + E_j(1-\cos\varphi_j) \tag{14}$$

is the Hamiltonian of a Josephson junction,

$$H_L = \frac{p_L^2}{2m} + \frac{1}{2L}\Phi_L^2, \tag{15}$$

which is equivalent to a harmonic oscillator, and

$$H_{\rm int} = \zeta n p_L \tag{16}$$

is the coupling term,

$$E_c^{(j)} = \frac{2e^2}{C_j}, \quad \frac{1}{m} = \frac{C_j+C_c}{C_jC_c}, \quad \zeta = -\frac{2e}{C_j}. \tag{17}$$

Since the LC circuit can be referred to as an equivalent harmonic oscillator, as is mentioned above, we can obtain the quantum mechanical Hamiltonian $\hat{H}_L$ by replacing p_L, Φ_L by the corresponding operators, with $[\hat{p}_L, \hat{\Phi}_L] = -i\hbar$. Thus, the Hamiltonian $\hat{H}_L$ can be written in the standard form

$$\hat{H}_L = \hbar\omega(\hat{c}^\dagger\hat{c} + 1/2), \tag{18}$$

where $\omega = \sqrt{\dfrac{C_j+C_c}{LC_jC_c}}$ is the characteristic frequency of the harmonic oscillator, and

$$\begin{aligned}\hat{c}^\dagger &= \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{\Phi}_L - i\hat{p}_L),\\ \hat{c} &= \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{\Phi}_L + i\hat{p}_L)\end{aligned} \tag{19}$$

are the bosonic operators and satisfy the relation $[\hat{c}, \hat{c}^\dagger] = 1$.

According to Feynman's view[6] that 'a bound pair acts as a Bose particle', we are naturally led to provide the bosonic operator model to quantize the classical Hamiltonian in Eq. (14). As is mentioned above, the excess charge $2en$ should be quantized, so we let

$$n \to \hat{\mathcal{N}} \equiv \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b} \tag{20}$$

be the excess Cooper-pair number operator, where $\hat{a}^\dagger$ is the positive charge Bose creation operator and $\hat{b}^\dagger$ is the negative charge Bose creation operator. To embody the number-phase quantization scheme we introduce the entangled state $|\eta\rangle$[8−12]

$$|\eta\rangle = \exp\Big[-\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^* b^\dagger + a^\dagger b^\dagger\Big]|00\rangle\Big\rangle, \tag{21}$$

where $\eta = |\eta|e^{i\varphi_j}$, and $|00\rangle$ is the two-mode vacuum state. The $|\eta\rangle$ state is constructed based on the idea of quantum entanglement of Einstein, Podolsky and

Rosen.[13] Using $[\hat{a},\hat{a}^\dagger]=[\hat{b},\hat{b}^\dagger]=1$, we can find that $|\eta\rangle$ obeys the eigenvector equations

$$(\hat{a}-\hat{b}^\dagger)|\eta\rangle=\eta|\eta\rangle,\quad (\hat{b}-\hat{a}^\dagger)|\eta\rangle=-\eta^*|\eta\rangle. \tag{22}$$

The set of $|\eta\rangle$ makes up a complete representation

$$\int\frac{d^2\eta}{\pi}|\eta\rangle\langle\eta|=1. \tag{23}$$

We can construct the bosonic phase operator for Josephson junction[14–16]

$$e^{i\hat{\varphi}_j}=\sqrt{\frac{\hat{a}-\hat{b}^\dagger}{\hat{a}^\dagger-\hat{b}}},\quad e^{-i\hat{\varphi}_j}=\sqrt{\frac{\hat{a}^\dagger-\hat{b}}{\hat{a}-\hat{b}^\dagger}},$$
$$\cos\hat{\varphi}_j=\frac{1}{2}(e^{i\hat{\varphi}_j}+e^{-i\hat{\varphi}_j}), \tag{24}$$

because in the $|\eta\rangle$ representation, $e^{i\hat{\varphi}_j}$ behaves a phase

$$e^{i\hat{\varphi}_j}|\eta\rangle=e^{i\varphi_j}|\eta\rangle,\quad e^{-i\hat{\varphi}_j}|\eta\rangle=e^{-i\varphi_j}|\eta\rangle. \tag{25}$$

Note $[\hat{a}^\dagger-\hat{b},\hat{a}-\hat{b}^\dagger]=0$, so they can reside in the same square root. It then follows $\hat{\varphi}_j=\dfrac{1}{2i}\ln\dfrac{\hat{a}-\hat{b}^\dagger}{\hat{a}^\dagger-\hat{b}}$, $\hat{\varphi}_j|\eta\rangle=\varphi_j|\eta\rangle$. From Eq. (22) we can derive

$$\begin{aligned}\hat{\mathcal{N}}|\eta\rangle&\equiv(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b})|\eta\rangle\\&=[\hat{a}^\dagger(\eta+\hat{b}^\dagger)-\hat{b}^\dagger(\hat{a}^\dagger-\eta^*)]|\eta\rangle\\&=|\eta|(\hat{a}^\dagger e^{i\varphi_j}+\hat{b}^\dagger e^{-i\varphi_j})|\eta\rangle\\&=-i\frac{\partial}{\partial\varphi_j}|\eta\rangle.\end{aligned} \tag{26}$$

Therefore

$$\langle\eta|[\hat{\varphi}_j,\hat{\mathcal{N}}]=\Big[\varphi_j,i\frac{\partial}{\partial\varphi_j}\Big]\langle\eta|=-i\langle\eta|\to[\hat{\varphi}_j,\hat{\mathcal{N}}]=-i,$$
$$[\cos\hat{\varphi}_j,\hat{\mathcal{N}}]=i\sin\hat{\varphi}_j,\ [\sin\hat{\varphi}_j,\hat{\mathcal{N}}]=-i\cos\hat{\varphi}_j, \tag{27}$$

which embodies the number-phase quantization. As a consequence of Eqs. (21)–(27), the classical Hamiltonian in Eq. (14) is quantized as

$$\hat{H}_j=E_c^{(j)}\hat{\mathcal{N}}^2+E_j(1-\cos\hat{\varphi}_j), \tag{28}$$

and the coupling term in Eq. (16) is quantized as

$$\hat{H}_{\rm int}=i\zeta\sqrt{m\hbar\omega/2}\hat{\mathcal{N}}(\hat{c}^\dagger-\hat{c}). \tag{29}$$

From the Heisenberg equation and Eq. (27) we deduce

$$2e\frac{d}{dt}\hat{\mathcal{N}}=\frac{2e}{i\hbar}[\hat{\mathcal{N}},\hat{H}_j]=\frac{2eE_j}{\hbar}\sin\hat{\varphi}_j\equiv\hat{I}_j, \tag{30}$$
$$\begin{aligned}\frac{d\hat{\varphi}_j}{dt}&=\frac{1}{i\hbar}[\hat{\varphi}_j,\hat{H}_j+H_{\rm int}]\\&=\frac{1}{i\hbar}\boldsymbol{E}_c^{(j)}[\hat{\varphi}_j,\mathcal{N}^2]\\&\quad+\frac{1}{i\hbar}\boldsymbol{i\zeta}\sqrt{m\hbar\omega/2}(\hat{c}^\dagger-\hat{c})[\hat{\varphi}_j,\hat{\mathcal{N}}]\\&=-\frac{1}{\hbar}[2E_c^{(j)}\mathcal{N}+i\zeta\sqrt{m\hbar\omega/2}(\hat{c}^\dagger-\hat{c})]\\&=\frac{2e\hat{U}_j}{\hbar}.\end{aligned} \tag{31}$$

From above we can see that the Josephson current equation keeps the same form but the voltage equation is affected because of the capacitor coupling, since $\dfrac{1}{m}=\dfrac{C_j+C_c}{C_jC_c}$ as shown in Eq. (17). Comparing Eq. (31) with $\dot{\varphi}_j=\dfrac{2e}{\hbar}u_j$ one can see that the effective voltage drop across the Josephson junction is

$$\hat{U}_j=-\frac{E_c^{(j)}\hat{\mathcal{N}}}{e}-\frac{i\zeta\sqrt{m\hbar\omega/2}}{2e}(\hat{c}^\dagger-\hat{c}). \tag{32}$$

In the following we reveal a squeezing mechanism involved in the time evolution of a Josephson junction when extra energy is applied to the junction. For this purpose, we consider the work performed by the Josephson current through the junction in a short time interval Δt,

$$\begin{aligned}U_jI_j\Delta t=&-\frac{2eE_j\Delta t}{\hbar}\Big[\frac{E_c^{(j)}\hat{\mathcal{N}}}{e}\\&+\frac{i\zeta\sqrt{m\hbar\omega/2}}{2e}(\hat{c}^\dagger-\hat{c})\Big]\sin\hat{\varphi}_j.\end{aligned} \tag{33}$$

From Eq. (33) we can see that when an extra energy (say, light radiation) is applied to the Josephson junction, the corresponding Hamiltonian can be expressed in the following form:

$$\hat{\mathcal{H}}'=\lambda\Big[\frac{E_c^{(j)}\hat{\mathcal{N}}}{e}+\frac{i\zeta\sqrt{m\hbar\omega/2}}{2e}(\hat{c}^\dagger-\hat{c})\Big]\sin\hat{\varphi}_j, \tag{34}$$

where λ is a coupling constant, and the constant term $-\dfrac{2eE_j}{\hbar}$ is absorbed into λ. Taking $\mathcal{H}'$ in the interaction picture and according to the equation of motion in this picture, we derive

$$\frac{d}{dt}\sin\hat{\varphi}_j=\frac{1}{i\hbar}[\sin\hat{\varphi}_j,\hat{\mathcal{H}}']=-\frac{\lambda E_c^{(j)}}{2e\hbar}\sin2\hat{\varphi}_j, \tag{35}$$

$$\frac{d}{dt}\cos\hat{\varphi}_j=\frac{1}{i\hbar}[\cos\hat{\varphi}_j,\hat{\mathcal{H}}']=\frac{\lambda E_c^{(j)}}{e\hbar}\sin^2\hat{\varphi}_j. \tag{36}$$

It then follows

$$\frac{d}{dt}\tan\frac{\hat{\varphi}_j}{2}=\frac{d}{dt}\Big(\frac{1-\cos\hat{\varphi}_j}{\sin\hat{\varphi}_j}\Big)=-\frac{\lambda E_c^{(j)}}{e\hbar}\tan\frac{\hat{\varphi}_j}{2}, \tag{37}$$

which has the solution

$$\tan\frac{\hat{\varphi}_j}{2}=\exp\Big\{-\frac{\lambda E_c^{(j)}}{e\hbar}t\Big\}\tan\frac{\hat{\varphi}_j(0)}{2}. \tag{38}$$

This solutions shows the evolution behaviour of phase, which is squeezed according to the damping factor $\exp\left\{-\frac{\lambda E_c^{(j)}}{e\hbar}t\right\}$. Thus an extra energy $\mathcal{H}'$ applied to the junction will cause the squeezing of $\tan\frac{\hat{\varphi}_j}{2}$. Since the Josephson current is caused by the phase-difference between two superconductors, the squeezing of $\tan\frac{\hat{\varphi}_j}{2}$ certainly accompanies the variation of extra number of Cooper-pairs on the island. Due to the coupling capacitance C_c, the inductance and Josephson junction should be considered in its totality. Thus, for the inductance, though it is not directly radiated by light, we can still obtain the induction-voltage operator equation

$$\hat{u}_L = \frac{d}{dt}\hat{\Phi}_L = \frac{1}{i\hbar}[\hat{\Phi}_L, \hat{\mathcal{H}}'] = \frac{\lambda\zeta}{2e}\sin\hat{\varphi}_j, \qquad (39)$$

which shows how the phase $\hat{\varphi}_j$ in the Josephson junction affects induction-voltage across the inductance through the coupling capacitance.

In summary, using the Hamilton dynamic approach we have constructed a Hamiltonian operator for the mesoscopic LC circuit including a Josephson junction. The entangled state representation is used to propose the Cooper-pair number-phase quantization and the Hamiltonian operator for the whole system. The Josephson current operator equation and the voltage operator equation are obtained. We have shown how the capacitance coupling affects the voltage across the Josephson junction.

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