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Numerical Study on Fokker-Planck Equation of Bistable System Driven by Colored Noise

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A finite difference method is used to solve a Fokker-Planck equation of bistable system with Landau potential. The detailed dynamical relaxation process in the case of large correlation time is manifested via the phenomena including the saddle point appearance, the hole formation and distortion. The method is used to obtain the stationary solutions of Fokker-Planck equation of bistable system driven by rather weak noise.

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In many publications it has already been demonstrated that the matrix-continued-fraction method (MCF) is a powerful tool to get the numerical solutions of Fokker-Planck equation (FPE).[1-4] However, in the weak noise case, this method becomes inpractical, because the convergence property of the MCF is poor then. Nevertheless, the weak noise situation is particularly realistic in practical systems.[5,6] In most cases, the FPEs do not satisfy detailed balance and then defy analytical treatment. Therefore, a numerical study of FPE in the weak noise is very important. The problem of colored noise with large correlation time and weak strength is one of the most typical examples of this kind.

This letter tries to numerically solve two-dimensional FPE representing a one-dimensional system driven by colored noise with the form as

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[(x - x^3 + y)P\right] + \gamma\frac{\partial}{\partial y}(yP) + \varepsilon\frac{\partial^2 P}{\partial x^2} \quad , \tag{1}$$

where ε represents the noise strength, while $1/\gamma$ is the correlation time of the colored noise.[5,6] Obviously, two stable points of the deterministic stationary solution of Eq. (1) are located at $(+1, 0)$ and $(-1, 0)$, respectively.

Fokker-Planck equation is a kind of parabolic partial differential equation, which may be transferred to a finite difference equation with Crank-Nicolson implicit mode. As in computational fluid mechanics and heat transfer,[7] the upstream scheme is used instead of the central difference scheme to overcome the difficulty of divergence, which may appear in weak noise case. In addition, the fifty power function approximation introduced by Patana[8] is adopted in this work as well to decrease the computing time while keeping enough precision.

We assume the initial distribution be Gaussian, and $x=\pm 1.5$; $y=\pm 1$ be infinite boundary. That means the boundary condition is assumed as

$$P(x,y,t)=0 \quad \text{as} \quad x=\pm 1.5 \quad \text{or} \quad y=\pm 1 \,. \tag{2}$$

The solution can be obtained as an initial value problem by relaxation iterative method.

Figures 1(a)–1(d) are the figures of the contour lines of probability distributions, which demonstrate transient process of FPE with $\varepsilon=0.01$, $\gamma=0.2$. Figure 1(a) is the initial distribution, Figs. 1(b)–1(d) are the results issued under iteratives with 40, 260 and 700 periods respectively. As in these Figures, the probability density outside of the contours is practically equal to zero; the distorted contours in Fig. 1(b) is due to the boundary condition (2), which, however, does not considerably influence the distribution around the attractors. Actually, as time increases, the influence of the boundary cannot be felt as shown in Figs. 1(c) and 1(d).

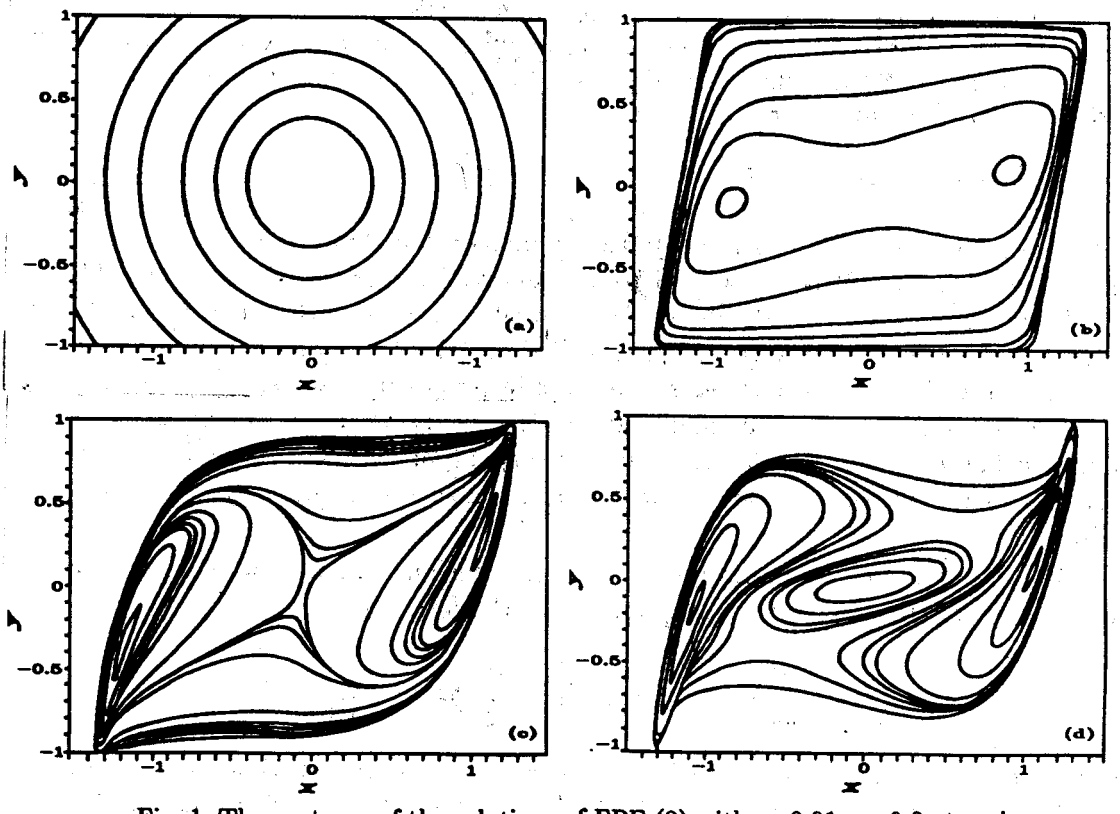

Fig. 1: The contours of the solutions of FPE (2) with $\varepsilon=0.01$, $\gamma=0.2$ at various iteration stages. The iteration times are (a) the initial distribution be Gaussian with standard deviation 0.6, (b) 40, (c) 260, (d) 700.

It is interesting to see the dynamical relaxation process, how the initial Gaussian distribution centered at the origin evolves to a distribution with a probability hole in the center, as shown in Figs.1(a)–1(d). There are essentially two stages. First, the initial distribution centered at the origin [see Fig. 1(a)] evolves away from the origin along x-axis as shown in Fig. 1(b) and forms two separated peaks; then the two new peaks

are driven toward points $(\pm 1, 0)$ clockwise, the peak height increases and approaches to the stable value. In the first stage, one can see the first saddle formed at the origin and in the second stage one can find that the single saddle at the origin bifurcates to two saddles leaving the origin. Figure 1(c) shows the probability distribution profile when the single saddle has just bifurcated to two side saddles. It is just in the second stage that a probability hole appears at the origin as shown in Fig. 1(d). Figures 2(a)–2(d) are the intersector curves of probability distributions corresponding to Figs. 1(a)–1(d) along the y-axis, from which one can see the saddle bifurcation process more clearly. Though many numerical works report the existence of the probability hole at the deterministic saddle,[2–4] it is a new result that we find how the hole forms during the time evolution. The existence of the saddle at the origin in an intermediate time regime is an interesting feature of the transient process.

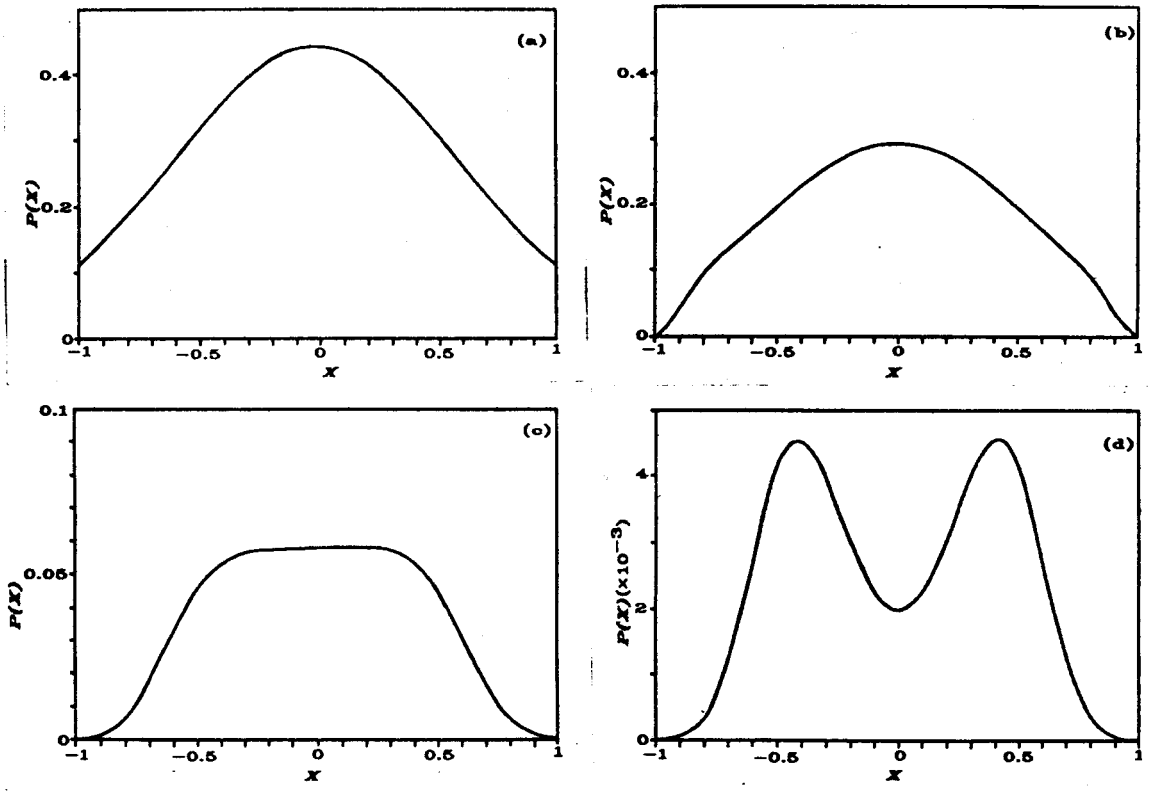

Fig. 2: The intersector curves of the probability distributions along y-axis. The cases of (a)–(d) correspond to the case of Figs. 1(a)–1(d), respectively.

Figures 3(a) and 3(b) are the contour diagrams of the stationary solutions in the cases of ε=0.001; γ=0.1 and 0.2, respectively. In all these parameter conditions, the probability hole exists obviously. However, as γ increases (i.e. the correlation time of colored noise decreases), the hole is continuously distorted, and contracted. At γ=0.8 the hole disappears. In that case, two separated saddles merge and form a single saddle again.

This method can be applied to resolve FPE with different drift and diffusion terms,

for instance

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\Big[C_1(x,y,t)P\Big] - \frac{\partial}{\partial y}\Big[C_2(x,y,t)P\Big] + D_1\frac{\partial^2 P}{\partial x^2} + D_2\frac{\partial^2 P}{\partial y^2} \quad . \tag{3}$$

The disadvantage of the finite difference method for solving FPE is that the CPU time consuming for reaching a stationary solution is very large.

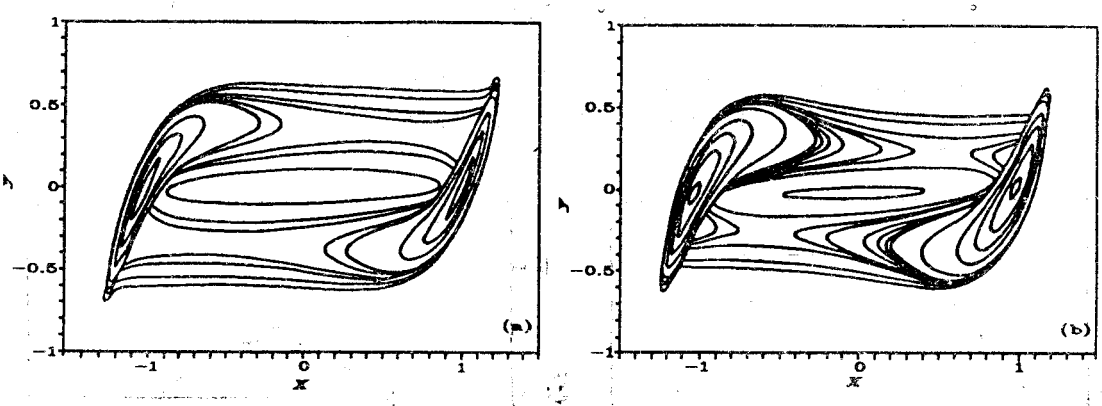

Fig. 3: The contours of stationary distributions of ε=0.001 (a) γ=0.1, (b) γ=0.2.

The main new results of the present letter are the following: (1) We use an effective numerical arithmetic to solve FPEs with small diffusion coefficients and large correlation time which are beyond the parameter regimes ever detected; (2) By time dependent solution we show, clearly for the first time, the dynamic process of the central probability hole under large correlation time, which enables us understand better the curious phenomenon, that a probability hole appears at a deterministic saddle.

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