A Geometry Model for Tortuosity of Streamtubes in Porous Media with Spherical Particles *

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A geometry model for tortuosity of tortuous streamtubes in porous media with spherical particles is proposed based on the assumption that some particles in a porous medium are unrestrictedly overlapped and hence of different configurations. The proposed model is a function of porosity with no empirical constant imposed on it. The model predictions are found to be in good agreement with the available experimental data.

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The characteristics of material transportation in porous media, such as the hydraulic conductivity, electrical conductivity and diffusivity have steadily received much attention[1−16] due to many unresolved problems in science and engineering applications. These macroscopic transport parameters are usually related to the tortuosity of tortuous flow paths/streamtubes followed by transported materials. However, it is known that the tortuous path a transported material follows is microscopically very complicated. Therefore, the tortuosity for tortuous path in porous media is conventionally determined by experiments[3] and the experimental data are then correlated as correlations with one or more empirical constants. In addition, the tortuosity for flow path/streamtube in porous media can also be determined by numerical methods. Rigby and Gladden[5] applied three different numerical methods (the whole object averaging method, diffusion flux method and random walk method) to calculate the tortuosity, and the tortuosity for diffusivity of gases/fluids was expressed as a series. Koponen *et al.*[6] applied the lattice gas (LG) cellular automaton method to solve numerically a creeping flow of Newtonian incompressible fluid in a two-dimensional porous substance constructed by randomly placed rectangles of equal size and with unrestricted overlap. Their numerical results were correlated as empirical correlations with one or two empirical constants, which have no physical meaning. Wang *et al.*[7] derived an approximate correlation between the non-Darcy coefficient tensor, tortuosity tensor and permeability tensor. However, no explicit expression for the tortuosity tensor has been presented. Moldrup *et al.*[8] applied the diffusion-based tortuosity analysis to describe the relation between tortuosity and diffusivity. They found that liquid-phase tortuosity is strongly soil-type dependent and liquid-phase geometry, while gas-phase tortuosity is less soil-type dependent and related to gas-phase connectivity (connectivity of air-filled pores). This has important implications towards understanding and modelling diffusive and convective transports in the gaseous and liquid phases of unsaturated soil. However, the tortuosity relies on measurement of diffusivity.

In order to achieve a better understanding of the mechanisms for tortuosity of tortuous flow paths/streamtubes through porous media, an analytical model for tortuosity is desirable and this is a challenging task. Westhuizen and du Plessis[4] derived a tortuosity model based on the ratio of the fluid volume enclosed by the unit cell to the minimum pore cross-sectional flow area in the unit cell. However, this model is valid only to the porous media with square cylinders without overlapping. Recently, Yu and Li[13] proposed a simple geometry model for tortuosity of tortuous flow path in porous media with randomly placed and unrestrictedly overlapped cubes. This tortuosity model is expressed as a function of porosity and there is no empirical constant in the model. However, this model is only suitable for porous media with cubic particles, and it is not applicable to porous media with spherical particles. So far, no analytical model has been reported for tortuosity of flow paths/streamtubes in porous media with spherical particles to the best of our knowledge. This work is devoted to deriving an analytical model for tortuosity of tortuous flow paths/streamtubes in porous media with spherical particles.

The tortuosity is often defined by[1,2]

$$\tau = L_e/L, \tag{1}$$

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where L_e and L are the actual length of flow path and the straight length or thickness of a sample along the macroscopic pressure gradient, respectively.

A correlation between the average tortuosity of flow path/streamtube can be given by[3]

$$\tau = 1 + C\ln(1/\phi), \tag{2}$$

which was obtained by the experiments on flow through beds packed with spherical particles, where C $(= 0.41)$ is the empirical/fitting constant obtained by fitting the experiment data, and ϕ is the porosity.

Koponen *et al.*[6] showed an expression for tortuosity,

$$\tau^2 = \tau_1\tau_{-1}, \tag{3}$$

where

$$\tau_1 = \frac{\int_V \tilde{\tau} v dV}{\int_V v dV}, \quad \tau_{-1} = \frac{\int_V v dV}{\int_V v dV/\tilde{\tau}},$$

$\tilde{\tau} = \tilde{\tau}(r)$ is the ratio of the length of streamlines passing through the point r to the thickness of a sample, $v = |v(r)|$ is the tangential velocity of fluid at point r, and V is the volume of a porous sample. Theoretically, as long as the analytical expressions for $\tilde{\tau}$ and v are known, we can obtain the tortuosity from Eq. (3). Unfortunately, analytical expressions for $\tilde{\tau}$ and v are not available. Since Eq. (3) cannot be used directly to calculate the tortuosity, Koponen *et al.*[6] calculated the tortuosity by applying the LG method for velocity fields and for tortuosity. Here we focus on deriving an analytical model for tortuosity by applying the geometrical method, and the model predictions will be compared with the existing experimental data.

By averaging over the tortuosities of all the possible tortuous flow paths/streamtubes, in principle, we can find the average tortuosity for the tortuous flow paths/streamtubes in porous media by

$$\tau = \frac{1}{N}\sum_i \tau_i, \tag{4}$$

where N is the total number of flow paths/streamtubes and τ_i is the tortuosity for the ith flow path/streamtube. Because of the diversity of flow paths/streamtubes, we must choose some representative flow paths/streamtubes and average over them to obtain the average tortuosity.

Figures 1(a), 1(b), 1(c) and 2 display the possible configurations for a Newtonian incompressible fluid flowing through porous media of two-dimensional spherical particles. The particles in Fig. 1 are arranged in an equilateral-triangle form, while those in Fig. 2 are arranged in a square form. The dashed rectangles as shown in Fig. 1 are the representative unit cells, R is the radius of spherical particles, and d is the gap size between particles.

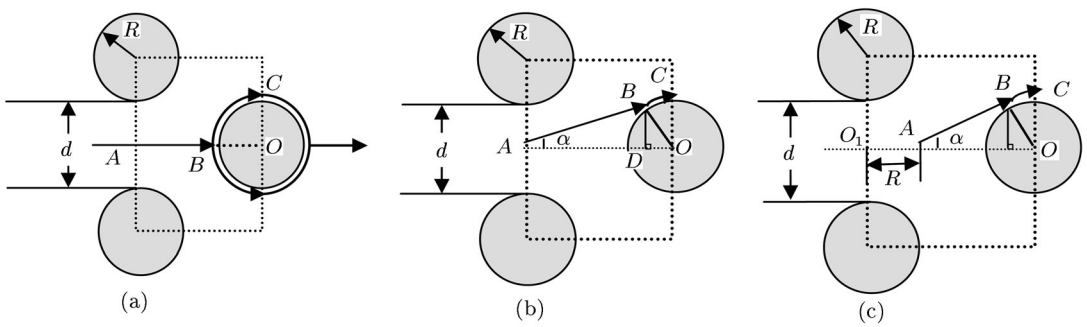

Fig. 1. Representative flow paths/streamtubes in porous media with spherical particles arranged in equilateral triangles.

For a unit cell as shown in Fig. 1, the total volume is $V_t = \sqrt{3}\cdot(2R+d)^2/2$, and the total pore volume in the unit cell is $V_P = \sqrt{3}(2R+d)^2/2 - \pi R^2$. Thus, the porosity is given by

$$\phi = \frac{V_P}{V_t} = 1 - \frac{\pi}{\frac{\sqrt{3}}{2}\left(2+\frac{d}{R}\right)^2}. \tag{5}$$

From Eq. (5), we obtain $\dfrac{d}{R} = \sqrt{\dfrac{2\pi}{\sqrt{3}(1-\phi)}} - 2$. Let the ratio $d/R = P$, then

$$P = \frac{d}{R} = \sqrt{\frac{2\pi}{\sqrt{3}(1-\phi)}} - 2. \tag{6}$$

For Fig. 1(a), the actual length of flow path is $l_{AB} + l_{\overset{\frown}{BC}} = \sqrt{3}(2R+d)/2 - R + \pi R/2$, and the straight length is $l_{AO} = \sqrt{3}(2R+d)/2$. Since the boundary layer thickness is very thin on the surface of particle, the arc length $l_{\overset{\frown}{BC}}$ is approximately $\pi R/2$. Therefore, according to the definition Eq. (1), we have the tortu-

osity for Fig. 1(a) as

$$\tau_{1a} = \frac{l_{AB} + l_{\overset{\frown}{BC}}}{l_{AO}} = 1 + \frac{\sqrt{3}(\pi - 2)}{6 + 3P}. \tag{7}$$

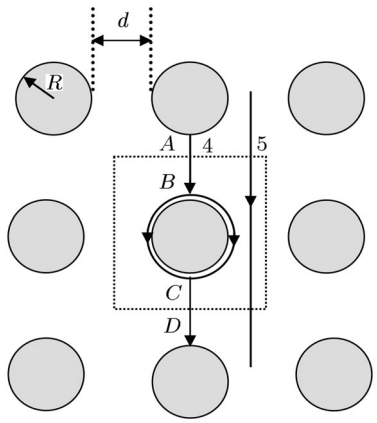

Fig. 2. Representative flow paths/streamtubes in porous media with spherical particles arranged in squares.

For Fig. 1(b), the actual length of flow path is $l_{AB} + l_{\overset{\frown}{BC}}$, the streamline AB is tangent to the particle, the triangle AOB is approximately a right-angled triangle. The length of the streamline AB is $l_{AB} = \sqrt{l_{AO}^2 - l_{BO}^2} = \sqrt{2R^2 + \frac{3}{4}d^2 + 3Rd}$, $l_{\overset{\frown}{BC}} = \alpha R$, α is expressed as $\alpha = \arcsin\frac{l_{BO}}{l_{AO}} = \arcsin\left[\frac{\sqrt{3}}{2}(2+p)\right]^{-1}$. Thus, we obtain the tortuosity for Fig. 1(b) as

$$\tau_{1b} = \frac{l_{AB} + l_{\overset{\frown}{BC}}}{l_{AO}} = \frac{\sqrt{2 + \frac{3}{4}P^2 + 3P} + \arcsin\left[\frac{\sqrt{3}}{2}(2+p)\right]^{-1}}{\sqrt{3}(2+P)/2}. \tag{8}$$

Since particles in actual porous media are randomly distributed, this means that some particles overlap each other, while the others do not. Therefore, the arc BC in Fig. 1(b) (the second configuration of Fig. 1(b) is assumed) might be sheltered from each other because of the overlapping, and this overlapping leads to some streamtubes skipping up the arc BC. Then, the tortuosity is controlled by flow paths/streamtubes in the *pore scale* in this situation. Therefore, the actual length of flow path is l_{AB} (not $l_{AB} + l_{\overset{\frown}{BC}}$), the corresponding straight length is l_{AD} (not l_{AO}). Figure 1(c) is another similar situation, in which both the streamline length and the arc length BC are overlapped/skipped up, thus the tortuosity for the flow paths (the second configuration of Fig. 1(b) and configuration as shown in Fig. 1(c)) is in the *pore scale*, and it is expressed as

$$\tau_{bc} = l_{AB}/l_{AD} = 1/\cos\alpha. \tag{9}$$

For Fig. 1(b) (the second configuration of Fig. 1(b)), $\alpha = \arcsin\left[\frac{\sqrt{3}}{2}(2+p)\right]^{-1}$; for Fig. 1(c), $\alpha = \arcsin\left[\frac{\sqrt{3}}{2}(2+p) - 1\right]^{-1}$. Therefore, according to the definition of Eq. (1), we obtain the tortuosity as

$$\tau'_{1b} = \frac{1}{\sqrt{1 - \left[\frac{1}{\sqrt{3}(2+p)/2}\right]^2}}, \tag{10}$$

$$\tau_{1c} = \frac{1}{\sqrt{1 - \left[\frac{1}{[\sqrt{3}(2+p)/2} - 1\right]^2}}, \tag{11}$$

respectively.

According to Eq. (4), for Figs. 1(a), 1(b) and 1(c), the averaged tortuosity is

$$\tau_1 = \frac{\tau_{1a} + \tau_{1b} + \tau'_{1b} + \tau_{1c}}{4}. \tag{12}$$

In addition to the arrangements as shown in Fig. 1, there might be other arrangements, such as the square arrangement as shown in Fig. 2. For Fig. 2, we have the total volume of the unit cell (see the dashed square in Fig. 2) $V_t = (2R + d)^2$, the total pore volume in the unit cell is $V_P = (2R + d)^2 - \pi R^2$, thus we obtain $\phi = V_P/V_t = 1 - \pi[2 + d/R]^{-2}$. Then, we have $\frac{d}{R} = \sqrt{\frac{\pi}{1-\phi}} - 2$. Here, we use P_1 to replace d/R, i.e.

$$P_1 = \sqrt{\frac{\pi}{1-\phi}} - 2. \tag{13}$$

For streamline 4 in Fig. 2, $l_{AB} = l_{CD} = d/2$, the arc length $l_{\overset{\frown}{BC}} = \pi R$, thus

$$\tau_4 = \frac{l_{AB} + l_{\overset{\frown}{BC}} + l_{CD}}{l_{AB} + 2R + l_{CD}} = 1 + \frac{\pi - 2}{P_1 + 2}. \tag{14}$$

For streamline 5 in Fig. 2, the actual length is equal to the straight length of flow path in the unit cell, so its tortuosity is 1, i.e.

$$\tau_5 = 1. \tag{15}$$

It is expected that the streamline is tortuous near the particle and is almost straight far away from the particle. It is also expected that the fraction of the straight flow path increases with the increasing porosity, but the fraction of the tortuous flow path decreases with the increasing porosity. Therefore, we take the weighted average tortuosity as follows:

$$\tau_2 = \left(1 - \frac{d}{d+R}\right)\tau_4 + \frac{d}{d+R}\tau_5, \tag{16}$$

where $d/(d+R)$ is defined as the weighted factor. Equation (16) indicates that when $d \to 0$, $d/(d+R) \to 0$ and $\tau_2 \to \tau_4$. This means that only tortuous streamtubes exist. When $d \to \infty$, $d/(d+R) \to 1$ and $\tau_2 \to \tau_5 = 1$, the gap d between the particles is infinity and the straight streamtubes prevail the fluid field. These results are consistent with the general observation.

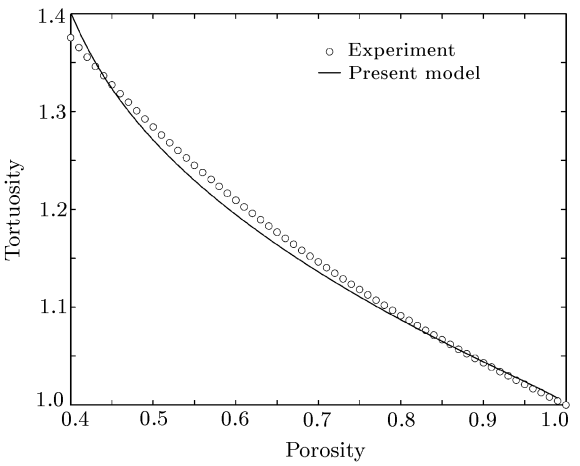

Fig. 3. A comparison between the present model predictions and the experimental correlation Eq. (2).

Now we can obtain the averaged tortuosity for flow paths/streamtubes in porous media with spherical particles arranged in different configurations by averaging over Eqs. (12) and (16):

$$\tau_{av} = (\tau_1 + \tau_2)/2. \tag{17}$$

Equation (17) is an approximate expression for tortuosity of tortuous flow paths/streamtubes in porous media with spherical particles. This equation is expressed as a function of porosity and there is no empirical constant. Though Eq. (2) looks simple, it has an empirical constant which has no physical meaning and the mechanism of the empirical constant is ignored. The proposed model looks somewhat complicated, while it is helpful for understanding the physical mechanism for tortuosity of tortuous flow paths/streamtubes in porous media with spherical particles.

Figure 3 compares the present model predictions with the available experimental correlation of Eq. (2). It is seen that the present model presents good agreement with the existing experimental correlation when the porosity is higher than 0.4, with relative error less than 2%. The porosity of 0.4 is approximately the minimum porosity (the maximum porosity is close to 1 near the wall) in the bulk/core region in packed beds.[17] It is also worth pointing out that the correlation Eq. (2) shows that the porosity can approach to zero, which is inconsistent with practical situation with spherical particles. However, the present model does not have such a weakness. This can be seen from the expression $\dfrac{d}{R} = \sqrt{\dfrac{\pi}{1-\phi}} - 2$ since d/R must be positive. The minimum porosity, which cannot be zero for porous media with spherical particles, can be found by assuming $d/R = 0$ (meaning all particles touch each other). Therefore, the correlation Eq. (2) may present unreasonable results at lower porosity.

In conclusion, we have proposed a geometry model for tortuosity of tortuous flow paths/streamtubes in porous media with spherical particles by assuming that some particles are unrestrictedly overlapped and hence of different configurations. The proposed model is a function of porosity with no empirical constant imposed on it. The proposed model might be helpful to understand the physical mechanism of tortuosity of tortuous flow paths/streamtubes in porous media with spherical particles. The model predictions are found to be in good agreement with the available experimental data when the porosity is higher than 0.4.

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