# Explicit Protocol for Deterministic Entanglement Concentration *

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*We present an explicit protocol for extraction of an EPR pair from two partially entangled pairs in a deterministic fashion via local operations and classical communication. This protocol is constituted by a local measurement described by a positive operator-valued measure (POVM), one-way classical communication, and a corresponding local unitary operation or a choice between the two pairs. We explicitly construct the required POVM by the analysis of the doubly stochastic matrix connecting the initial and the final states. Our scheme might be useful in future quantum communication.*

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Quantum entanglement plays a major role in quantum information processing, such as teleportation,[1] dense coding,[2] secret key distribution,[3] and quantum computation.[4] All these applications require that two distant parties, usually called Alice and Bob, share maximally entangled EPR pairs. In practical situations, however, we have to deal with imperfect entangled states, including mixed entangled states and partially entangled pure states. Fortunately, various entanglement purification schemes have been proposed,[5−19] by which Alice and Bob can extract a certain number of almost perfectly entangled pairs from a larger number of less-entangled pairs using local operations and classical communication (LOCC). When the initial less-entangled states are pure, this purification is specially termed entanglement concentration. Recently, entanglement purification[20] and entanglement concentration[21,22] of polarization-entangled photon pairs have been experimentally realized.

This study only concerns the entanglement concentration, for which several protocols have been proposed. The Schmidt projection method[5] extracts EPR pairs from many identical entangled pairs via collective measurements on many particles. Lo and Popescu[6] derived the maximum probability with which an EPR pair can be obtained by concentrating a single entangled pair, which is achieved by the Procrustean method.[5] The schemes proposed by Bose *et al.*,[7] Zhao *et al.*,[8] and Yamamoto *et al.*[9] have the advantage of being experimentally feasible. Note that all these schemes are probabilistic for finite number of entangled pairs. Now suppose that we are supplied with finite (even small) number of entangled pairs, and we are not allowed to run the risk of failing in quantum communication. In this scenario, we have to try to achieve entanglement concentration without gambling, i.e. in a deterministic fashion. This question has been considered by Morikoshi and Koashi[11,12] who obtained an explicit condition using Nielsen's theorem[23] for extracting a maximally entangled pair from two partially entangled pairs with unit probability.

With all those efforts, the protocol for deterministic entanglement concentration still has not been constructed. One may argue that the proof of Nielsen's theorem is constructive, but for practical applications Nielsen's scheme is too complex since it involves several rounds of individual measurements and classical communication, while it has been shown that such a sequence of operations can reduce to one involving only a single measurement and one-way classical communication.[6] In this Letter, we present an explicit protocol for deterministic extraction of a maximally entangled pair from two partially entangled pairs, which may be in different states but satisfy certain conditions. This protocol is constituted by Alice's local measurement described by a positive operator-valued measure (POVM), one-way classical communication from Alice to Bob, and Bob's corresponding local unitary operation or choice between the two pairs. Our main task is to construct the required POVM following the line of Jensen and Schack[24] As far as we know, no simpler or more direct methods exist for constructing the POVM.

Suppose that Alice and Bob share two partially entangled pairs of two-level particles (qubits), the states of which can be written in Schmidt decomposition as

$$|\psi\rangle_{AB} = \sqrt{a}|00\rangle_{AB} + \sqrt{1-a}|11\rangle_{AB},$$
$$|\varphi\rangle_{A'B'} = \sqrt{b}|00\rangle_{A'B'} + \sqrt{1-b}|11\rangle_{A'B'}, \qquad (1)$$

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where the Schmidt coefficients $a$ and $b$ are positive real numbers and should be known in advance. Further, suppose that particles $A$ and $A'$ belong to Alice, and particles $B$ and $B'$ to Bob. Without loss of generality, we can set $1/2 < a \leq b < 1$. Note that for any bipartite pure state $|\psi\rangle_{AB}$, $\rho_\psi^A = \mathrm{tr}_B(|\psi\rangle_{ABAB}\langle\psi|) = a|0\rangle_{AA}\langle 0| + (1-a)|1\rangle_{AA}\langle 1|$, $\rho_\psi^B = \mathrm{tr}_A(|\psi\rangle_{ABAB}\langle\psi|) = a|0\rangle_{BB}\langle 0| + (1-a)|1\rangle_{BB}\langle 1|$, so the eigenvalues of $\rho_\psi^A$ and $\rho_\psi^B$ are identical, namely $a$ and $1-a$. It is shown in Refs. [11,12] that using LOCC, Alice and Bob can deterministically transform the two pairs into

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}),$$
$$|\phi\rangle_{A'B'} = |00\rangle_{A'B'}, \tag{2}$$

under the condition $ab \leq 1/2$. In fact, the four particles can be regarded as a bipartite system, with Alice's two particles $A$ and $A'$ as one part and Bob's two particles $B$ and $B'$ as the other part. The vectors of eigenvalues of the reduced density operators of the initial and the final states are $\lambda_{\psi\otimes\varphi}^T = (ab, (1-a)b, a(1-b), (1-a)(1-b))^T$ and $\lambda_{\Phi^+\otimes\phi}^T = (1/2, 1/2, 0, 0)^T$, respectively, where the elements have been arranged in descending order. It is clear that if $ab \leq 1/2$, then $\lambda_{\psi\otimes\varphi}$ is majorized by $\lambda_{\Phi^+\otimes\phi}$. From Nielsen's theorem, it follows that the initial state can be deterministically transformed into the final state via LOCC.

A square matrix is called a doubly stochastic matrix if it has non-negative real entries and each row and column sums to 1. Mathematically, that $\lambda_{\psi\otimes\varphi}$ is majorized by $\lambda_{\Phi^+\otimes\phi}$ is equivalent to the existence of a doubly stochastic matrix $D$ such that $\lambda_{\psi\otimes\varphi} = D\lambda_{\Phi^+\otimes\phi}$,[25] which, written explicitly, reads

$$\begin{pmatrix} ab \\ (1-a)b \\ a(1-b) \\ (1-a)(1-b) \end{pmatrix} = D \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}. \tag{3}$$

Using the algorithm of Ref. [25], we find the corresponding doubly stochastic matrix $D$ through lengthy but straightforward calculation, the non-zero elements of which are

$$D_{11} = \frac{a+b-ab-1/2}{a+b-2ab},$$
$$D_{12} = \frac{(2a-1)(2b-1)(1/2-ab)}{a+b-2ab},$$
$$D_{13} = \frac{a(1-b)(1/2-ab)(2a-1)(2b-1)}{(a+b-2ab)(1/2-a+ab)},$$
$$D_{14} = \frac{(1-a)b(1/2-ab)}{(a+b-2ab)(1/2-a+ab)},$$
$$D_{22} = 2(1-a)b,$$
$$D_{23} = \frac{2ab(1-a)(1-b)}{1/2-a+ab},$$
$$D_{24} = \frac{(2a-1)(2b-1)/2}{1/2-a+ab},$$
$$D_{32} = 2a(1-b),$$
$$D_{33} = 1-2a(1-b),$$
$$D_{41} = \frac{1/2-ab}{a+b-2ab},$$
$$D_{42} = \frac{(a+b-ab-1/2)(2a-1)(2b-1)}{a+b-2ab},$$
$$D_{43} = \frac{a(1-b)(a+b-ab-1/2)(2a-1)(2b-1)}{(a+b-2ab)(1/2-a+ab)},$$
$$D_{44} = \frac{(1-a)b(a+b-ab-1/2)}{(a+b-2ab)(1/2-a+ab)}, \tag{4}$$

and $D_{21} = D_{31} = D_{34} = 0$. There is a result known as Birkhoff's theorem,[26,25] which states that a doubly stochastic matrix $D$ can be written as a convex combination of permutation matrices, $D = \sum_i p_i P_i$, where $p_i > 0$, $\sum_i p_i = 1$, and $P_i$ are permutation matrices. Using the algorithm of Ref. [24], a tedious calculation shows that the matrix $D$ can be decomposed to be

$$D = D_{13}P_1 + D_{22}I + D_{42}P_3 + D_{12}P_4 + (D_{44} - D_{22})P_5 + D_{14}P_6 + D_{43}P_7, \tag{5}$$

with

$$P_1 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix},$$
$$P_4 = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 1&0&0&0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix},$$
$$P_6 = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}, \quad P_7 = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}, \tag{6}$$

and $I$ is the identity matrix. We now substitute Eq. (5) into Eq. (3). Considering the specific form of $\lambda_{\Phi^+\otimes\phi}$, when the matrix $D$ acts on it, the permutation matrices $P_1$ and $P_6$, $P_3$ and $P_4$, $P_5$ and $P_7$ have the same effects, respectively. Therefore we have

$$\begin{pmatrix} ab \\ (1-a)b \\ a(1-b) \\ (1-a)(1-b) \end{pmatrix} = p_1 \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + p_2 \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} + p_3 \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix} + p_4 \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}, \tag{7}$$

where

$$\begin{aligned} p_1 &= D_{22} = 2(1-a)b, \\ p_2 &= D_{44} - D_{22} + D_{43} = 2a - 1, \\ p_3 &= D_{42} + D_{12} = (2a-1)(2b-1), \\ p_4 &= D_{13} + D_{14} = 1 - 2ab. \end{aligned} \tag{8}$$

We observe that the columns of the matrix in Eq. (7) are the results of $\lambda_{\phi^+\otimes\phi}$ acted on by the permutation matrices $I$, $P_5$, $P_3$, $P_6$, respectively.

Now a trick will be applied. We define four diagonal matrices $A_j$ $(j = 1, 2, 3, 4)$ by taking each column of the matrix in Eq. (7) and dividing the $i$th entry (in the $i$th row) with the $i$th element of the left-hand side vector in Eq. (7), and multiplying with $p_j$. Thus

$$\begin{aligned} A_1 &= p_1 \mathrm{diag}\left(\frac{1/2}{ab}, \frac{1/2}{(1-a)b}, 0, 0\right) \\ &= \mathrm{diag}\left(\frac{1-a}{a}, 1, 0, 0\right), \\ A_2 &= p_2 \mathrm{diag}\left(\frac{1/2}{ab}, 0, \frac{1/2}{a(1-b)}, 0\right) \\ &= \mathrm{diag}\left(\frac{2a-1}{2ab}, 0, \frac{2a-1}{2a(1-b)}, 0\right), \\ A_3 &= p_3 \mathrm{diag}\left(\frac{1/2}{ab}, 0, 0, \frac{1/2}{(1-a)(1-b)}\right) \\ &= \mathrm{diag}\left(\frac{(2a-1)(2b-1)}{2ab}, 0, 0, \frac{(2a-1)(2b-1)}{2(1-a)(1-b)}\right), \\ A_4 &= p_4 \mathrm{diag}\left(0, 0, \frac{1/2}{a(1-b)}, \frac{1/2}{(1-a)(1-b)}\right) \\ &= \mathrm{diag}\left(0, 0, \frac{1-2ab}{2a(1-b)}, \frac{1-2ab}{2(1-a)(1-b)}\right). \end{aligned} \tag{9}$$

It is straightforward to verify that each $A_j$ is a positive matrix and that the sum of $A_j$ is the identity matrix, namely $\sum_j A_j = I$. Thus $\{A_j\}$ define a POVM. Let us see now what happens when the POVM is applied locally to the initial state in Eq. (1). Suppose that Alice performs the POVM measurement on her two particles $A$ and $A'$. Then the post-measurement states corresponding to the measurement results $j$ are (up to normalization) $\sqrt{A_j}|\psi\rangle_{AB} \otimes |\varphi\rangle_{A'B'}$, with probabilities $p_1$, $p_2$, $p_3$, $p_4$, respectively (the form of Eq. (3) implies that $A_j$ are matrices of corresponding operators with respect to the basis $\{|00\rangle_{AA'}, |10\rangle_{AA'}, |01\rangle_{AA'}, |11\rangle_{AA'}\}$, with the basis vectors not lexicographically ordered). Normalization of these states yields

$$\begin{aligned} &\frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}) \otimes |00\rangle_{A'B'}, \quad j = 1 \\ &|00\rangle_{AB} \otimes \frac{1}{\sqrt{2}}(|00\rangle_{A'B'} + |11\rangle_{A'B'}), \quad j = 2 \\ &\frac{1}{\sqrt{2}}(|0000\rangle_{ABA'B'} + |1111\rangle_{ABA'B'}), \quad j = 3 \\ &\frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}) \otimes |11\rangle_{A'B'}, \quad j = 4. \end{aligned} \tag{10}$$

Note that tracing out $B$ and $B'$, the vectors of eigenvalues of the reduced density operators are just the same as the corresponding columns of the matrix in Eq. (7). To bring the states in Eq. (10) to the state in Eq. (2), Alice performs one of the unitary operations $U_j = I$, $P_5$, $P_3$, $P_6$, conditional upon her measurement results $j = 1, 2, 3, 4$ (clearly Alice's measurement and unitary operation can be combined into one single measurement described by $\{U_j\sqrt{A_j}\}$), and informs Bob of her measurement results. Then Bob performs the same operation as Alice, which completes the entanglement conversion. However, for our purpose of entanglement concentration, things are even simpler. If Alice's measurement result is $j = 1$ or 4, Alice and Bob do not need to perform any operation. What they need to do is to simply discard the particle pair $A'B'$ and keep the pair $AB$ for future use. If $j = 2$, they discard the pair $AB$ and keep the pair $A'B'$. If $j = 3$, Alice and Bob happen to acquire a valuable four particle GHZ state. Only in this case do they both need to perform corresponding operation $P_3$ to obtain a two particle EPR pair. Rewriting $P_3$ in the lexicographically ordered basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ we obtain

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This is just a Control-NOT gate. For the case of $j = 3$, an alternative way is for Alice and Bob to adopt the last part of the scheme in Ref. [8]. From the above analyses one can see that no matter what measurement result is obtained, entanglement concentration is always realized. In other words, entanglement concentration can be realized with unit success probability, which is just what deterministic entanglement concentration means.

The most important point of our scheme is the POVM measurement described by Eq. (9), which can be implemented with a quantum circuit and a projective measurement.[26] To do this, we introduce a four-state ancillary system $C$ having an orthonormal basis $\{|j\rangle_C, j = 1, 2, 3, 4\}$, e.g. two qubits. Define an operator $U$ acting on particles $A, A', C$ by

$$U|\psi\rangle_{AB} \otimes |\varphi\rangle_{A'B'}|1\rangle_C = \sum_j \sqrt{A_j}|\psi\rangle_{AB} \otimes |\varphi\rangle_{A'B'}|j\rangle_C. \tag{11}$$

Evidently, $U$ preserves inner products, therefore can be extended to a unitary operator. Thus $U$ can be implemented with a quantum circuit.[26] After that, a projective measurement is performed on the ancillary system $C$. From Eq. (11), if the result is $j$, the pairs $AB$ and $A'B'$ are projected onto the corresponding state in Eq. (10) with probability $p_j$. We will not give a detailed description of the quantum circuit here as a Letter.

In summary, we have presented an LOCC protocol for extracting a maximally entangled EPR pair from two partially entangled pairs. Different from other probabilistic protocols, this protocol makes it possible for us to perform entanglement concentration without gambling, i.e. in a deterministic fashion. Therefore it is especially suitable for situations requiring high reliability. The protocol is constituted by one party's local measurement described by a positive operator-valued measure (POVM), one-way classical communication, and the other party's corresponding local unitary operation or choice between the two pairs. Based on the analysis of the doubly stochastic matrix connecting the initial and the final states, we explicitly construct the required POVM and show that it can be implemented with a quantum circuit and a projective measurement. This scheme might be useful in future quantum communication.

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