Limit Cycles near Stationary Points in the Lorenz System *

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The limit cycles in the Lorenz system near the stationary points are analysed numerically. A plane in phase space of the linear Lorenz system is used to locate suitable initial points of trajectories near the limit cycles. The numerical results show a stable and an unstable limit cycle near the stationary point. The stable limit cycle is smaller than the unstable one and has not been previously reported in the literature. In addition, all the limit cycles in the Lorenz system are theoretically proven not to be planar.

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The Lorenz equations[1] are one of the most famous dynamic systems in studies of chaos, which are a three-dimensional system of ordinary differential equations, i.e.

$$\begin{aligned}\dot{x}_1 &= -\sigma x_1 + \sigma x_2,\\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3,\\ \dot{x}_3 &= x_1 x_2 - b x_3.\end{aligned} \tag{1}$$

Here we take $\sigma = 10$ and $b = 8/3$ that are the most commonly used values in the studies of the Lorenz system. The system we adopt has a natural symmetry $(x_1, x_2, x_3) \to (-x_1, -x_2, x_3)$. Three stationary points exist for $r > 1$,

$$(C^0) \quad x_1 = x_2 = x_3 = 0, \tag{2a}$$

$$(C^+) \quad x_1 = x_2 = \sqrt{b(r-1)}, \quad x_3 = r - 1, \tag{2b}$$

$$(C^-) \quad x_1 = x_2 = -\sqrt{b(r-1)}, \quad x_3 = r - 1. \tag{2c}$$

The system is strongly nonlinear and the chaotic behaviour of the Lorenz equations is found by numerical simulations for the first time in nonlinear dynamic systems.

There also exist limit cycles in the Lorenz system, which are of great significance in mathematics and physics. Limit cycles can organize the global dynamics of the system, can help to study the route to chaos,[2−4] and can control the chaotic state.[5,6] A general theorem for establishing the existence of a limit cycle in the Lorenz system was developed by Coomes *et al.*[7] A method with the Lie derivatives for estimation of domains with limit cycles and finding surfaces with the traces of all cycles was proposed by Krishchenko[8] and Babloyantz *et al.*[9] The numerical study of limit cycles in the Lorenz system by Sparrow[10] and Strogatz[11] shows that C^+ is encircled by an unstable cycle for $13.926\cdots < r < 24.74\cdots$.

In this Letter, we extend the study of the limit cycles in the Lorenz system. The initial points near C^+ are located on a plane with the numerical results showing one stable and one unstable limit cycles near C^+. The structure near C^- is similar to the structure near C^+ in symmetry. In addition, all the limit cycles in the Lorenz system are proven not to be planar but only to be curves in space.

First of all, we prove that any limit cycle in the Lorenz system can not be planar. It is supposed that a limit cycle in the Lorenz system is on the plane with the general equation

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 = 0, \tag{3}$$

where a_1, a_2, a_3 and a_4 are the constants. The derivative of Eq. (3) is

$$a_1 \dot{x}_1 + a_2 \dot{x}_2 + a_3 \dot{x}_3 = 0. \tag{4}$$

From Eqs. (1) and (4), a space quadratic surface is obtained to satisfy the equation

$$\begin{aligned}a_3 x_1 x_2 - a_2 x_1 x_3 + (a_2 r - a_1 \sigma) x_1 \\ + (a_1 \sigma - a_2) x_2 - a_3 b x_3 = 0.\end{aligned} \tag{5}$$

The limit cycle on the plane described by Eq. (3) must satisfy the intersection of Eqs. (3) and (5). The possible intersection types are discussed in two cases: (1) $a_2 = a_3 = 0$, Eq. (5) describes a plane, and the possible intersection of Eqs. (3) and (5) is a straight line; (2) a_2 and a_3 are not zero simultaneously.

Equation (5) describes a space quadratic surface with three orthogonal invariants

$$I_1 = 0, \ I_2 = -(a_2^2 + a_3^2)/4 \neq 0, \ I_3 = 0. \tag{6}$$

The characteristic equation of Eq. (5) is

$$\rho^3 - I_1 \rho^2 + I_2 \rho - I_3 = 0, \tag{7}$$

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with three roots

$$\rho_1 = -\rho_2 = \sqrt{a_2^2 + a_3^2}/2 \neq 0, \quad \rho_3 = 0. \tag{8}$$

By orthogonal transformation of the coordinates, Eq. (5) is reduced to

$$\rho_1(x_1'^2 - x_2'^2) + px_3' + q = 0, \tag{9}$$

where p and q are the constants. The possible intersection of any plane and the quadratic surface described by Eq. (9) is a hyperbola, parabola or straight line.[12]

From cases (1) and (2), the possible intersection of the plane described by Eq. (3) and the quadratic surface described by Eq. (5) should only be either a hyperbola, a parabola or a line, all of which are not closed curves and contradict that the limit cycle is a closed curve by definition. Consequently, the limit cycle in the Lorenz system can not be a planar line.

Next, a numerical method capturing limit cycles in the Lorenz system is shown here. The linearized Lorenz equations in the vicinity of C^+ in the $C^+y_1y_2y_3$ coordinate system which is a translation of the coordinate system $C^0x_1x_2x_3$ from C^0 to C^+ are

$$\begin{aligned} \dot{y}_1 &= -\sigma y_1 + \sigma y_2, \\ \dot{y}_2 &= y_1 - y_2 - \sqrt{b(r-1)}y_3, \\ \dot{y}_3 &= \sqrt{b(r-1)}y_1 + \sqrt{b(r-1)}y_2 - by_3. \end{aligned} \tag{10}$$

The Jacobian matrix of the linear Lorenz system in Eq. (10) for $\sigma = 10$ and $b = 8/3$ is

$$J = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & -\sqrt{(8/3)(r-1)} \\ \sqrt{(8/3)(r-1)} & \sqrt{(8/3)(r-1)} & -8/3 \end{bmatrix}. \tag{11}$$

The Jacobian matrix can also be written in form of the equations as

$$J = R\Lambda R^{-1}, \tag{12}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ is the eigenvalue matrix and R is the right eigenvector matrix; $\lambda_1 < 0$, λ_2 and λ_3 are conjugate complex numbers with the real part $\mathrm{Re}(\lambda_2) < 0$ for $13.926\cdots < r < 24.74\cdots$.

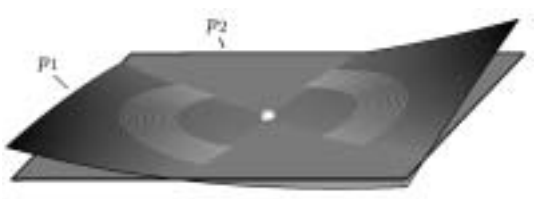

Fig. 1. Relative positions of p_1 and p_2. The central white ball indicates C^+. The white spiral trajectory is on p_1 with the part shown in the figure being above p_2. The limit cycles proven not to be planar are on p_1.

A surface with small curvature denoted by p_1 in the phase space of the Lorenz system contains C^+ and the limit cycles near C^+. A plane denoted by p_2 exists in the phase space of the linear Lorenz system contains C^+. Here p_1 is very close to p_2 (Fig. 1). According to linear stability analysis,[11] the trajectories with initial points located on p_2 will be quickly absorbed onto p_1, will then spiral on p_1, and will finally be absorbed into C^+ or fly away. The limits are found by initially placing points on the trajectories on p_2, with the trajectories then studied in sequence. The expression for p_2 will be obtained in the following.

Let

$$Q = \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & -i & i \end{bmatrix}, \tag{13}$$

and the transform matrix

$$P = QR^{-1}. \tag{14}$$

P is a real matrix. Perform a linear transform

$$Z = PY. \tag{15}$$

In the coordinate system $C^+z_1z_2z_3$, the linear Lorenz Eq. (10) becomes

$$\dot{Z} = BZ. \tag{16}$$

Here

$$B = \begin{bmatrix} \lambda_1 & & \\ & \mathrm{Re}(\lambda_2) & -\mathrm{Im}(\lambda_2) \\ & \mathrm{Im}(\lambda_2) & \mathrm{Re}(\lambda_2) \end{bmatrix}, \tag{17}$$

where $\mathrm{Im}(\lambda_2)$ represents the imaginary part of λ_2, p_2 in $C^+y_1y_2y_3$ is transformed to the plane $z_2 - z_3$ in $C^+z_1z_2z_3$ by P. Thus any vector on p_2 is transformed onto the plane $z_2 - z_3$ in $C^+z_1z_2z_3$ by P. Let

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}. \tag{18}$$

From Eq. (15), $\boldsymbol{v}$ satisfies

$$(p_{11}\ p_{12}\ p_{13}) \cdot \boldsymbol{v} = 0. \tag{19}$$

Thus the normal vector of p_2 in $C^+y_1y_2y_3$ is

$$\boldsymbol{n} = (p_{11}\ p_{12}\ p_{13})^T. \tag{20}$$

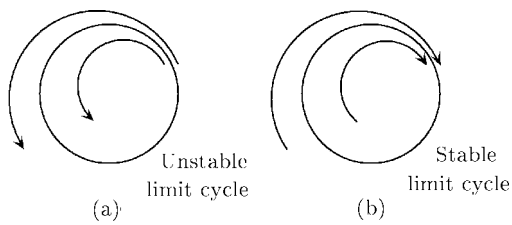

Fig. 2. Unstable and stable limit cycles.

Consequently, p_2 can be determined from the normal vector $\boldsymbol{n}$ and the stationary point C^+.

The search strategy for limit cycles is as follows. Let l be a line through C^+ on p_2. Locate point p_{t1} and point p_{t2} on l, with p_{t1} is located between C^+ and p_{t2}. Compute trajectory tr-1 with the initial point p_{t1} and trajectory tr-2 with the initial point p_{t2}. The two trajectories will soon be absorbed onto p_1, and will then spiral on p_1. Limit cycles are found by varying the locations of p_{t1} and p_{t2}. If tr-1 runs inwards and tr-2 runs outwards, an unstable limit cycle possibly exists between tr-1 and tr-2 [Fig. 2(a)]. On the contrary, if tr-1 runs outwards and tr-2 runs inwards, a stable limit cycle possibly exists between tr-1 and tr-2 [Fig. 2(b)]. Here p_{t1} and p_{t2} should very closely estimate the accurate properties and the position of the limit cycle. The limit cycles were searched for with different r values.

Function ODE45 of the commercial package Matlab (version 6.5) is used to compute the trajectories using the fourth-order five-stage automatic step-size Runge–Kutta method with double precision.

The computing strategy is used to find an unstable and a stable limit cycle and to calculate the radii and periods of the two limit cycles. The radius is the average distance from the point on the limit cycle to C^+. The computed results are presented in Table 1 with relative error tolerance (RelTol) 1×10^{-6} and absolute error tolerance (AbsTol) 1×10^{-9}. The results listed in Table 1 show that larger values of r reduce both the radius and the period of the unstable limit cycle, and that larger values of r increase the radius of the stable limit cycle and reduce the period of the stable limit cycle.

Table 1. Computed results.

r	Unstable limit cycle		Stable limit cycle	
	Radius	Period	Radius	Period
15.00	11.2	1.5352		
18.00	10.1	1.0220		
20.00	9.06	0.8746	1.31×10^{-4}	0.7143
22.00	7.35	0.7528	2.26×10^{-4}	0.6863
24.00	4.06	0.6783	6.99×10^{-4}	0.6591
24.50	2.34	0.6624	1.40×10^{-3}	0.6585
24.74	0.0730	0.6449	7.20×10^{-3}	0.6560

Figure 3 show the relative positions of the stable limit cycle and the unstable limit cycle in the cases $r = 24$ and $r = 24.5$. The results in Fig. 3 show that the stable limit cycle is within the unstable limit cycle for the same value of r in the two cases. Comparison of Fig. 3(a) with Fig. 3(b) shows that larger values of r reduce the unstable limit cycle, increase the stable limit cycle, and bring the two limit cycles closer to each other.

Figure 4 shows the structure of the Lorenz system near C^+ using the intuitive phase portrait and the bifurcation diagram. The results in Fig. 4 show that the unstable limit cycle is born at $r = 13.926\cdots$, while the stable limit cycle is born at $r = 19.3\cdots$, and larger values of r reduce the unstable limit cycle and increase the stable limit cycle. However, the stable limit cycle is always within the unstable limit cycle from birth to disappearance. The two limit cycles combine and disappear at $r = 24.74\cdots$.

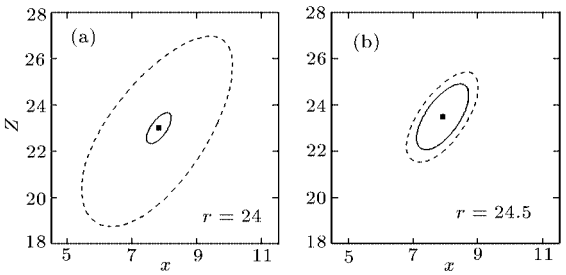

Fig. 3. Relative positions of the stable limit cycle (solid) and the unstable limit cycle (dashed) for (a) $r = 24$ and (b) $r = 24.5$. RelTol is 1×10^{-3} and AbsTol is 1×10^{-6}. Squares represent the position of C^+.

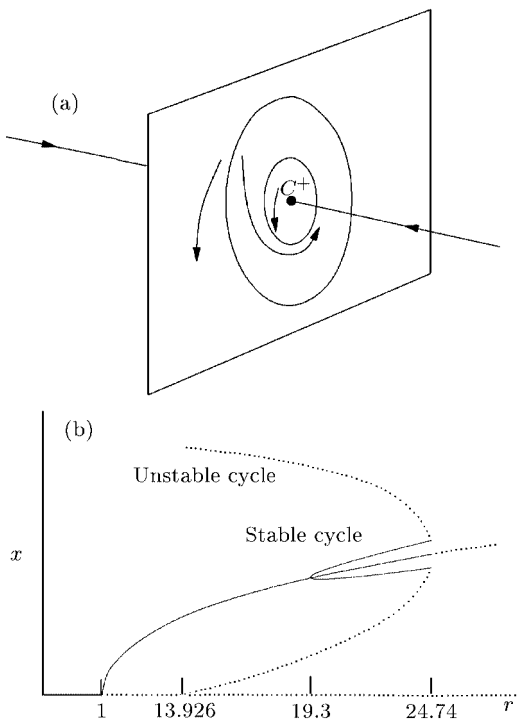

Fig. 4. Structure near C^+. (a) Manifold of limit cycles, (b) bifurcation diagram.

A new challenge comes up with the finding of the stable limit cycle. If the stable limit cycle and C^+ which is stable according to linear stability theory both absorb the trajectories between them, an unstable limit cycle must exist between the stable limit cycle and C^+, but the unstable structure is not found.

The properties of two limit cycles near the stationary point C^+ in the Lorenz system are studied by nu-

merically investigating trajectories with initial points on the plane p_2. Also all limit cycles in the Lorenz system are proven not to be planar curves but only to be curves in space. The numerical results show two limit cycles near C^+. One is an unstable limit cycle for $13.926\cdots < r < 24.74\cdots$,[10,11] while the other is a stable limit cycle for $19.3\cdots < r < 24.74\cdots$. The periods of both limit cycles decrease with r. Larger values of r reduce the unstable limit cycle radius and increase the stable limit cycle radius. However, the stable limit cycle is always within the unstable limit cycle. The two limit cycles combine and disappear at $r = 24.74\cdots$.

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